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Quaternionic Dirac oscillator

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Abstract

We construct an elementary quaternionic slow-fast Hamiltonian dynamical system with one formal control parameter and two slow degrees of freedom as half-integer spin in resonance 1:1:2 with two slow oscillators. Invariant under spin reversal and having a codimension-5 crossing of its fast Kramers-degenerate semi-quantum eigenvalues, our system is the dynamical equivalent of the spin-quadrupole model by Avron, Sadun, Segert, and Simon [Commun. Math. Phys. **124**(4), 595–627 (1989)], exhibiting non-Abelian geometric phases. The equivalence is uncovered through the equality of the spectral flow between quantum superbands and Chern numbers c_2 computed by Avron *et al.*

Keywords: Chern index, spectral flow, eigenvalue crossing, bands, edge and bulk states, nonlinear spin-oscillator, slow-fast resonance

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1 Introduction

Parametric families of quantum mechanical systems persist naturally as one of the principal research topics since the foundation of quantum science. Aiming at elementary phenomena, the analysis boils down to the study of possible degeneracies of the eigenvalues of real symmetric, Hermitian, or hyper-Hermitian traceless 2×2 matrices

$$H_\xi = \begin{pmatrix} -m & h \\ h^* & m \end{pmatrix} \text{ with } \det H_\xi = -m^2 - hh^*, \qquad (1.1a)$$

one of the most ubiquitous and universal mathematical problems [von Neumann and Wigner, 1929, Arnold, 1995]. While m is necessarily real, h can be either real, complex, or quaternionic. In the latter case, the isomorphism between Pauli matrices and unit quaternions suggests using

$$m = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} a + \mathrm{i}\, b & c + \mathrm{i}\, d \\ -c + \mathrm{i}\, d & a - \mathrm{i}\, b \end{pmatrix} \text{ with } (m, a, b, c, d) \in \mathbb{R}, \qquad (1.1b)$$

and considering *quaternionic* traceless 4×4 matrices with two doubly degenerate eigenvalues. These three basic possibilities make one of *Arnold's mathematical* $\mathbb{R}$-$\mathbb{C}$-$\mathbb{H}$ *trinities* [Arnold, 1997, 1999] illustrated in fig. 1. Since $\det H_\xi$ vanishes only in $m = h = 0$, the real codimension of the degeneracy of the eigenvalues of H_ξ, i.e., the number of real conditions to be met typically for these eigenvalues to cross, is 2, 3, and 5 in the $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ case, respectively.

Continuous, adiabatically slow evolution of parameters $\xi = (m, h)$ establishes *connections* on the Hilbert space of functions representing the states of quantum systems with Hamiltonian H_ξ in (1.1). This was known already to Herzberg and Longuet-Higgins [1963], but most eloquently, it has been demonstrated by Berry [1984]. He

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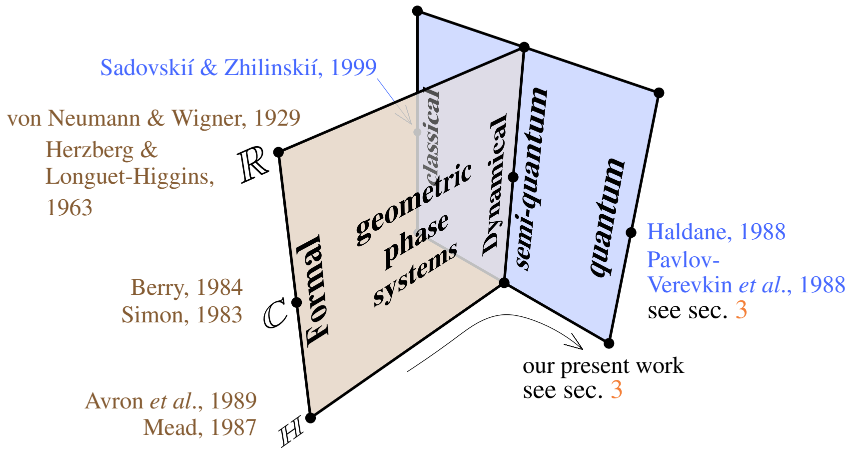

Figure 1: $\mathbb{R}$-$\mathbb{C}$-$\mathbb{H}$ trinity systems with geometric phases.

modeled the two eigenstates of H_ξ concretely as spin states $|\frac{1}{2}, \pm\frac{1}{2}\rangle$, whose interaction with magnetic field $\boldsymbol{B} = (B_1, B_2, B_3) =: \xi$ is described by the linear Hamiltonian $\boldsymbol{B} \cdot \hat{\boldsymbol{S}}$. This paradigm system has $\mathbb{C}$-type matrix representation (1.1a) with $2\,m = B_1$ and $2\,h = B_2 + \mathrm{i}B_3$. Considering the $\mathbb{C}$-bundle of eigenstates over any 2-sphere Δ surrounding $\xi = 0$ in the parameter space $\mathbb{R}^3_\xi$, Berry defined a connection on this bundle, now called commonly *Berry curvature* [Wilczek and Shapere, 1989], and demonstrated how this connection contributed to the phase accumulated by the eigenfunction while the latter was continued along cycles on Δ. The nontrivial contribution, or the *geometric phase*, signals the presence of the degeneracy at $0 \in \mathbb{R}^3_\xi$. Simon [1983] observed immediately that the associated curvature form is equivalent to the one used in the computation of the first Chern number c_1 of the bundle. This gives the complementary topological characteristics of the degeneracy that we exploit in our work.

Shortly after the seminal introduction of the geometric phase to the broad physics community by Berry [1984] and Simon [1983], important enhancements (fig. 1) were initiated by Haldane [1988] and Pavlov-Verevkin et al. [1988], who suggested, on their respective physical examples, quantum Hall effect and spin-orbital coupling, that the formal Berry phase setup can be intrinsically extended, if we assume that (at least some of) parameters ξ support an additional physical dynamical structure. In the midst of numerous applications, experimental observations, and interpretations that followed across very distant fields, from particle physics to classical waves, the simple "molecular" example [Pavlov-Verevkin et al., 1988] received no special appreciation. Attention was shifted towards more complex quasi-continuous spectra and larger potential scope of applications, notably in solid state. At the same time, the minimalism of [Pavlov-Verevkin et al., 1988] suggests the existence of an *elementary singularity* of Hamiltonian dynamical slow-fast systems with nontrivial geometric phase, whose Chern index can be manifested directly through its *spectral flow*. All other systems can be decomposed into families of such elementary singularities (sec. 3).

Another important development, the last but not the least to be mentioned, was the quaternionic generalization (fig. 1) of the Berry $\mathbb{C}$-system by Mead [1987] and Avron et al. [1988, 1989]. In their physical examples with half-integer spins, they revealed the particular Z_2 symmetry making the eigenvalues of (1.1b) constitute two inseparable *Kramers doublets* [Kramers, 1930, Wigner, 1932] as *spin-reversal* invariance[1] $\mathcal{T}_S$. Considering the $\mathbb{C}^2$-bundles which the doublets form over a sphere $\mathbb{S}^4 \subset \mathbb{R}^5_\xi$ surrounding the degeneracy point 0, Avron et al. [1989] computed the second Chern number c_2 characterizing the degeneracy at 0 in such $\mathcal{T}_S$-invariant quaternionic systems. This way, they completed one of the most intriguing Arnold's trinities [Arnold, 1997], see (8) in [Arnold, 1999] and the left edge of the graph in fig. 1. They also computed another fingerprint of the degeneracy, the non-Abelian geometric phase. While some physical systems with such phase have been already investigated, up until now, it remained unclear what elementary Hamiltonian dynamical analogues of the quaternionic models in [Mead, 1987, Avron et al., 1988, 1989] can be, and what quantum manifestations of their nontrivial c_2 index are. We aim at answering these fundamental questions (sec. 3) and completing the *dynamical triad* in fig. 1.

2 Elementary Hamiltonian slow-fast singularities

Dynamically parameterized geometric phase systems have been reviewed recently in ref. [Iwai et al., 2020]. We consider systems with minimal number of adiabatically slow control parameters ξ, which equals the codimension of the degeneracy of the eigenvalues λ of (1.1). Since in this case, the degeneracy occurs typically at an *isolated point* in the parameter space Σ, we will place it in $\xi = 0$, and work in its sufficiently small regular neighbourhood Σ_0. And finally, typical degeneracies are *conical*, with nonvanishing derivatives $\partial\lambda/\partial\xi$ at $\xi = 0$.

2a Formal and dynamical control parameters. Parameters of models by Berry [1984], Mead [1987], and Avron et al. [1988, 1989], and of similar systems can be changed in any imaginable/required way. We call such parameters and systems *formal* or *general*. Parameters in the Hamiltonian dynamical analogues of these models are of two kinds, formal α and dynamical (q, p). As their notation implies, the latter are canonical dynamical variables of the slow Hamiltonian dynamical system, which can serve as (local) coordinates on the slow classical phase space P. The total parameter space Σ becomes a product of P and the space of formal parameters α (fig. 2). It is natural to

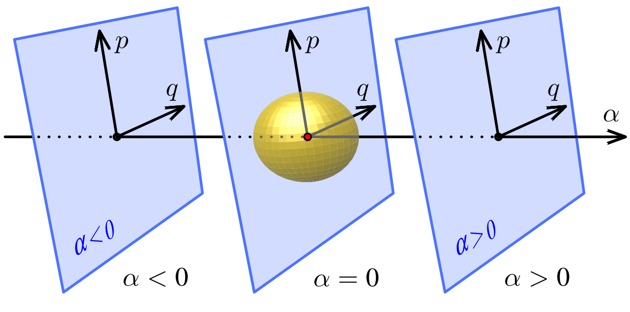

Figure 2: Total parameter space Σ of an elementary $\mathbb{C}$-system. The isolated degeneracy point 0 (red) of semi-quantum eigenvalues is surrounded by a sphere (yellow) which serves as the base space of the fiber bundle Δ. In the $\mathbb{H}$-case, the $(\boldsymbol{q}, \boldsymbol{p})$ plane and the sphere are four-dimensional. Compare to sec. 2.3 and fig. 12 of appendix 4 in [Iwai et al., 2020].

consider *maximally* dynamical $\mathbb{C}$ and $\mathbb{H}$-systems with α-space of minimal dimension

[1] While Mead [1987] and Avron et al. [1988, 1989] do not distinguish spin-reversal $\mathcal{T}_S$ and time-reversal $\mathcal{T}$ because they have no other dynamical variables than $\boldsymbol{S}$, we do, see eqs. (2.2) and (2.3) in sec. 2.

one and the number of slow degrees of freedom k equal to 1 and 2, respectively. The maximally dynamical $\mathbb{R}$ system, or dynamical diabolic point system, has no formal parameter. It can be seen[2] as a special member of the $\mathbb{C}$-family with $\alpha = 0$. Furthermore, working locally near $0 \in \Sigma$ implies flat open geometry with a line of formal parameter α and P a symplectic plane $\mathbb{R}^{2k}$.

2b **Spin systems as fast subsystems.** In the steps of Berry [1984] and Avron et al. [1988], we will use states with half-integer fixed spin S to construct concretely the respective two and four-level fast systems, and express the fast Hamiltonian in terms of components (S_1, S_2, S_3) of spin angular momentum $\boldsymbol{S}$. Specifically, we interpret (1.1b) in terms of the irreducible representation $\frac{3}{2}$ of $\mathrm{SU}(2)$. While many physical situations can be described effectively by a spin-$\frac{3}{2}$ multiplet, there exists a different, second realization of the quaternionic matrix (1.1b) using the fast basis of *two* spins, or more precisely, of spin $\boldsymbol{S}$ and pseudo-spin $\boldsymbol{S}'$, both of length $\frac{1}{2}$, i.e., the $(\frac{1}{2}, \frac{1}{2})$ representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. This model was introduced by Mead [1987, 1992] and Koizumi and Sugano [1995]. It has an additional first integral compensating the extra fast degree of freedom, and to the degeneracy of its bulk levels, it is similar to single-spin fast systems we analyze in this work.

2c **Bands, superbands, bulk and edge states.** The *semi-quantum* system with classical variables (q, p) and quantum operators $\hat{\boldsymbol{S}}$ is the dynamical equivalent of formal models by Berry [1984] and Avron et al. [1988, 1989]. Its quantum and classical limits (fig. 1, far end, left and right) retain α as their sole control parameter. This system has relatively few eigenvalues $\lambda_b : \Sigma \to \mathbb{R}$, $b \in \mathbb{N}$. In $\mathbb{H}$-systems, we use b to label Kramers-degenerate doublets of their semi-quantum eigenvalues. In the quantum limit, dynamical parameters become quantum operators $(\hat{q}, \hat{p})$, and eigenvalues λ_b turn into *bands* in $\mathbb{C}$-systems and Kramers degenerate *superbands* in $\mathbb{H}$-systems containing many discrete eigenstates. Commonly, at noncritical values of α, (super)bands are imagined as dense multiplets separated from each other by large energy gaps, i.e., the splittings within (super)bands are typically much smaller than those gaps. We like to stress that neither $\mathbb{C}$ nor $\mathbb{H}$-systems are invariant under time-reversal symmetry $\mathcal{T}$, and there are no specific degeneracies of their quantum levels. As we discuss further below, the presence of $\mathcal{T}$ incurs specific modifications of these elementary systems.

Observing the *spectral flow*, or the number of states transferred between (super)bands when α varies through the critical value $\alpha = 0$ corresponding to the semi-quantum eigenvalue degeneracy, one discovers that it equals the Chern number $c_k(\Delta)$. The states remaining within their (super)bands and those few being transferred are called *bulk* and *edge*, respectively. This terminology reflects the correspondence to more complex models in solid state, see [Iwai et al., 2020] and references therein. The classical limit is described by the one-parameter family of Hamiltonian equations of motion, which govern the evolution of both fast $\boldsymbol{S}$ and slow (q, p) dynamical variables. For $\mathbb{C}$-systems, this limit is related to Hamiltonian monodromy [Sadovskií and Zhilinskií, 1999], and, consequently, to the A_1 singularity [Sadovskií, 2016]. In this letter, we focus on the quantum limit of the $\mathbb{H}$-systems (fig. 1).

2d **Conical symmetry.** The most specific and important property of elementary Hamiltonian slow-fast singularities is their (local) *conical dynamical symmetry* $\mathrm{SO}(2)$. It originates in the fact that the degeneracy occurs at an isolated point, such as $\boldsymbol{q} = \boldsymbol{p} = 0$ for $\alpha = 0$. Concretely, we will consider simultaneous rotations of spin $\boldsymbol{S}$ about axis S_1

[2] Explicitly, note that matrix (1.1a) with $m = 0$ and $h = q + \mathrm{i}\, p$, which is the member of the $\mathbb{C}$-family, and real matrix (1.1a) with $m = q$ and $h = -p$ are conjugated under rotation $\boldsymbol{S} \mapsto (S_2, S_3, S_1)$.

and phase space rotations of $P=\mathbb{R}^{2k}$ generated by the flow of slow harmonic oscillator action I. In other words, our conical symmetry has momentum

$$J = S_1 + I. \tag{2.1}$$

J is first integral of the classical system, and its quantum analogue $\hat{J}$ commutes with quantum Hamiltonian $\hat{H}_\alpha$. The concrete choice of function $I : P \to \mathbb{R}$ will be justified later in sec. 3. We draw attention to two qualitatively different possibilities in $\mathbb{H}$-systems. Depending on the signs of oscillator frequencies, the image of I (and, consequently, J) can be either unbound or bound on one side.

2e **Spin-reversal.** No additional Lie symmetries should normally exist. We call *spin-reversal* the specific discrete symmetry operation

$$\mathcal{T}_S : (\boldsymbol{S}, \alpha, \boldsymbol{q}, \boldsymbol{p}) \mapsto (-\boldsymbol{S}, \alpha, \boldsymbol{q}, \boldsymbol{p}), \tag{2.2}$$

which is inherent to all $\mathbb{H}$-systems (sec. 1), and reserve time-reversal for operation

$$\mathcal{T} : (\boldsymbol{S}, \alpha, \boldsymbol{q}, \boldsymbol{p}) \mapsto (-\boldsymbol{S}, \alpha, \boldsymbol{q}, -\boldsymbol{p}). \tag{2.3}$$

The latter acts on *all* dynamical variables, and its presence is not essential. *Additional invariance*[3] under $\mathcal{T}$ is irrelevant to bands becoming degenerate. If $\mathcal{T}$ is present alongside $\mathcal{T}_S$, the *slow-reversal* $\mathcal{T} \circ \mathcal{T}_S$ makes the intersection at $q=p=0$ non-linear.

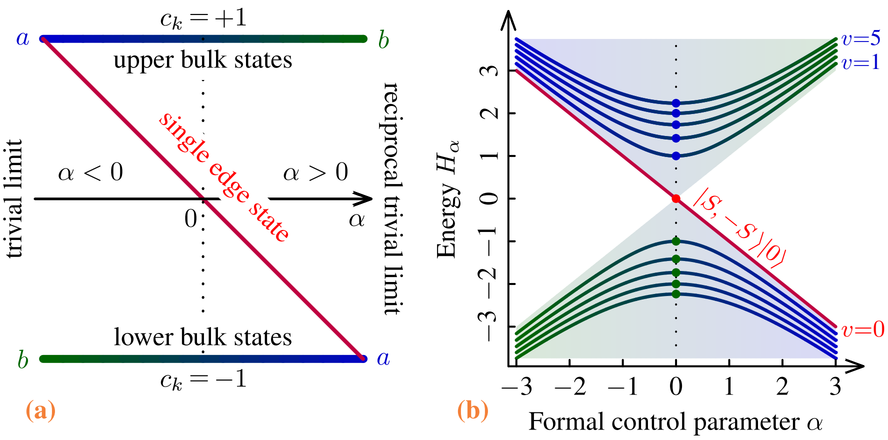

Figure 3: Correlation diagram (a) and spectrum (b) of the elementary $\mathbb{C}$ and $\mathbb{H}$ systems with spin-$\frac{1}{2}$ Dirac oscillator Hamiltonian (3.2a) and 1:1:2-resonant spin-$\frac{3}{2}$ quaternionic oscillator Hamiltonian (3.4), respectively. Bulk parts of (super)bands (a) and individual bulk states (b) of multiplicity 1 and v in the 1:1 and in the 1:1:2 system with $r=2$, respectively, are distinguished by solid blue-green lines. Solid red line represents the edge state. Light shade in (b) marks the image of the semi-quantum eigenvalues, and only levels with $v \le 5$ are displayed.

2f **Trivial limits and correlation diagram.** Any slow-fast system has two simple *uncoupled* or *trivial* reciprocal limits in which its fast and slow subsystems do not interact, and which are related to each other through energy reversal. Considering individual quantum eigenstates, we realize that some are unique and have to be redistributed

[3] The study of additional symmetries acting on slow variables requires further concretisation. We consider the slow subsystem locally as merely an abstract oscillator and analyze the most generic phenomenon. On the other hand, this subsystem can be a linearisation of a physical system with specific symmetries.

in order for the (super)band spectrum of the trivial limit to get reversed. In a continuous one-parameter family of systems connecting the two limits, this redistribution occurs only when its semi-quantum eigenvalues become degenerate. In elementary systems, the degeneracy is of the kind described above, and is characterized by the Chern number $c_k(\Delta)$, cf. sec. 1. Following individual eigenvalues of quantum Hamiltonians $H_\alpha(\hat{\boldsymbol{S}}, \hat{\boldsymbol{q}}, \hat{\boldsymbol{p}})$ as functions of formal parameter α, we compute the spectral flow[4]. The results can be represented as a *correlation diagram*, such as the one in fig. 3a. *It is our conjectured theorem that, to a sign convention, the thus obtained spectral flow for each (super)band* b *equals* $c_k(\Delta_b)$.

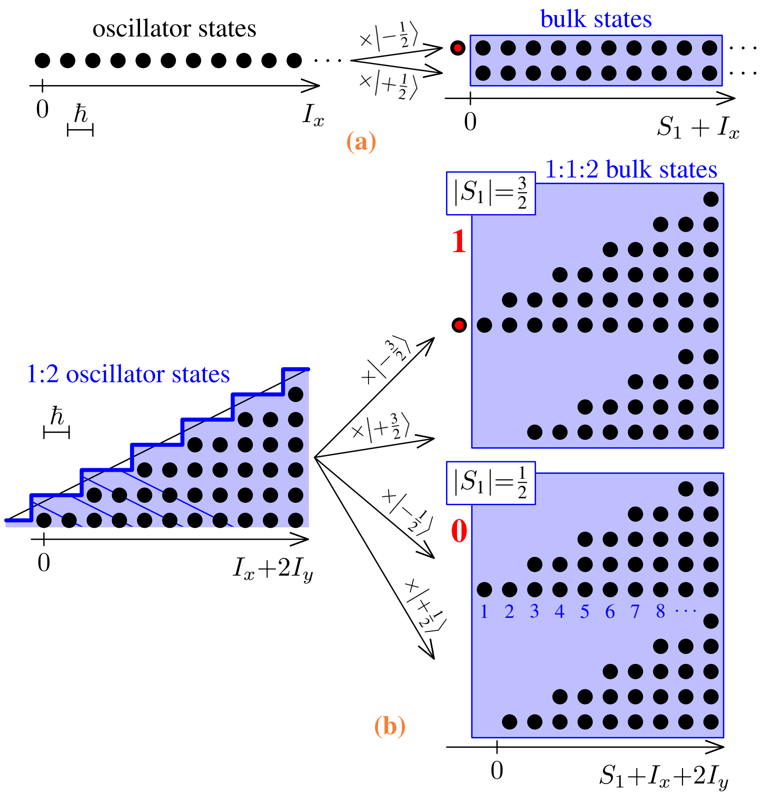

Figure 4: Uncoupled bases of (a) the two-band Dirac oscillator with $S=\frac{1}{2}$, and (b) the 1:1:2-resonant quaternionic Dirac oscillator with $S=\frac{3}{2}$ truncated at sufficiently large values of conical momentum j. Filled circles represent bulk (black) and edge (red) states; bulk state numbers $v = j + S$ are indicated for the lower superband of (b); the number of edge states is given in bold large red digits. See text for the explanation of the vertical axis in (b).

2g **Spectral flow and Chern numbers.** The conical $\mathrm{SO}(2)$ symmetry persists for all values of formal control parameter α and defines *completely* the distribution of the eigenstates in each (super)band over the irreducible representations of $\mathrm{SO}(2)$. In this aspect, $\mathrm{SO}(2)$ defines the structure of (super)bands and its modification, which occurs

[4] Unlike Atiyah and Singer [1968], we define spectral flows *for each (super)band* λ_b, and not just one flow for the entire system. In elementary $\mathbb{C}$ and $\mathbb{H}$ systems (sec. 2), where one single level transfers between two (super)bands, the spectral flow equals, to a sign, the topological charge 1: one band looses a state and has the flow of -1, while the other gets necessarily the flow of $+1$. Our detailisation becomes important in large-spin systems (sec. 3 and fig. 5b), where we consider many (super)bands with Chern numbers $c_k(\Delta_b)$.

after crossing the degeneracy point in the α-space. We denote quantum numbers of quantized momenta $\hat{J}$, $\hat{S}_1$, and $\hat{I}$ as j, S_1, and n. The $\mathbb{C}$-systems possess a one-dimensional oscillator slow subsystem with $(q, p) = (x, p_x)$, and I equals (to a sign) the standard harmonic oscillator action I_x (sec. 3). The $1{:}(\pm 1)$-weighted $\mathrm{SO}(2)$ symmetries are related by slow momentum reversal, and it suffices to understand the case of 1:1. The trivial eigenbasis of the spin-oscillator is the direct product of the $2S+1$-dimensional space of spin functions $|S, S_1\rangle$ and the "slow" Hilbert space of oscillator functions $|n\rangle$. In order to classify these states according to quantum number j of momentum (2.1), we shift the $\mathbb{N}_0$ lattice of oscillator states by S_1. In the two-band system with spin $\frac{1}{2}$ (fig. 4a), each band contains one state for every $j > -\frac{1}{2}$, while the state $|\frac{1}{2}, -\frac{1}{2}\rangle|0\rangle$ with *minimal* $j = -\frac{1}{2}$ (red dot) has no counterpart. When the spectrum of the two trivial bands gets reversed, this state is redistributed. Assuming that the described trivial limit and its reciprocal correspond to large negative and positive α, respectively, the spectral flow (cf. footnote 4) for individual bands $b = 1$ (upper) and $b = 2$ (lower) equals -1 and $+1$. In the elementary $\mathbb{C}$-system, these limits are connected so that the corresponding Chern numbers $c_1(\Delta_b)$ are equal to $+1$ and -1. The signs are fixed through the standard choice of matrix (1.1a) and of the corresponding Hamiltonian (3.2) in sec. 3.

$\mathbb{H}$-systems include a two-dimensional slow oscillator with dynamical variables (x, y, p_x, p_y), and their I equals (to a sign) $I_x \pm 2\, I_y$ (sec. 3). In order to explain the spectrum flow calculation, we consider first the two-superband system with weights 1:1:2 and spin $\frac{3}{2}$ (fig. 4b). Other cases are addressed later in sec. 3. The construction of the trivial-limit lattice of 1:2-oscillator states requires an additional first integral to serve as "height function" in fig. 4b. Locally, sufficiently near $(\boldsymbol{q}, \boldsymbol{p}) = 0$, we can use I_y and represent the lattice within a wedge, whose upper bound is given by the 1:2-sloped step-function (fig. 4b, left). The latter represents the total number of oscillator states with given value n of quantized 1:2-action I. We proceed similarly to the $\mathbb{C}$ case in fig. 4a, albeit now, we take the invariance under spin-reversal (2.2) into account and include all states with the same $|S_1|$ in one trivial superband (fig. 4b, right). We observe immediately that the system has a single edge state with $j = -\frac{3}{2}$, and therefore, by the above-mentioned conjectured theorem, its superbands have $c_2(\Delta)$ equal to ± 1 (fig. 3a).

It follows that elementary $\mathbb{C}$ and $\mathbb{H}$-systems have the same correlation diagram (fig. 3a) and *topological charge* $|c_k(\Delta)| = 1$. We also uncover the nature of the edge states (colored red in fig. 3 and 4), called so not only because of their transfer between bands [Iwai et al., 2020]. Their correspondence to the edge states in solids goes well beyond the mere fact that they participate in the spectral flow. These states have very specific strong localization. They are centered maximally at the degeneracy point $q = p = 0$ of the semi-quantum eigenvalues in the slow phase space P. We can think of them as "vortex states", which have, unlike bulk states, no easy classical and semi-classical description on P. In the $1{:}(\pm 1)$ and $1{:}(\pm 1){:}(\pm 2)$-systems, they are also highly localized in the fast (spin) variables near $\boldsymbol{S} = (\mp S, 0, 0)$. Fast localization of the missing states (sec. 3) of the $1{:}(\pm 1){:}(\mp 2)$-systems is different. Having minimal $|S_1| = \frac{1}{2}$, they can be seen as "near-equatorial".

3 Dirac oscillators

Replacing elements m, h, and h^* of (1.1a) by $\alpha \in \mathbb{R}$ and complex dynamical variables

$$a^\dagger = a^+ := \frac{q - \mathrm{i}p}{\sqrt{2}} = \bar{z}/\sqrt{2} \quad \text{and} \quad a = a^- := \frac{q + \mathrm{i}p}{\sqrt{2}} = z/\sqrt{2} \tag{3.1}$$

defines the semi-quantum matrix Hamiltonian

$$\mathsf{H}_\alpha(q,p) = \mathsf{H}_\alpha(x,p_x) = \alpha \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & a^\dagger \\ a & 0 \end{pmatrix} \tag{3.2a}$$

of the elementary $\mathbb{C}$-system with semi-quantum energies $\pm\sqrt{\alpha^2 + I_x}$ [Iwai et al., 2020] known as Dirac oscillator [Moshinsky and Szczepaniak, 1989]. Using basis spin functions $|\frac{1}{2}, -\frac{1}{2}\rangle$ and $|\frac{1}{2}, +\frac{1}{2}\rangle$ turns (3.2a) into a spin-oscillator Hamiltonian

$$H_\alpha(\hat{\boldsymbol{S}}, x, p_x) = \alpha\,\frac{\hat{S}_1}{S} + \frac{\hat{S}_+\, a + \hat{S}_-\, a^\dagger}{2\,S} \quad \text{with} \quad I_x = \tfrac{1}{2}\,(x^2 + p_x^2), \tag{3.2b}$$

which has 1:1-weighted conical symmetry $\mathrm{SO}(2)$ and $I = I_x$. It represents slow-fast resonance 1:1. Its isospectral 1:(−1) sibling with $I = -I_x$ is produced from (3.2b) under slow momentum reversal $\mathcal{T} \circ \mathcal{T}_S$.

Dynamical parameterization of the quaternionic matrix (1.1b) is defined similarly. In the concrete fast "canonical" spin-$\frac{3}{2}$ basis of [Avron et al., 1989, eq. (2.26), Definition 2.5, p. 605],

$$|\tfrac{3}{2}, \tfrac{3}{2}\rangle,\ |\tfrac{3}{2}, -\tfrac{3}{2}\rangle, |\tfrac{3}{2}, -\tfrac{1}{2}\rangle,\ |\tfrac{3}{2}, \tfrac{1}{2}\rangle, \tag{3.3}$$

matrix elements $\langle\frac{3}{2}, +S_1|H|\frac{3}{2}, -S_1\rangle$ of any $\mathcal{T}_S$-invariant operator H vanish, making diagonal 2×2 blocks $\pm m$ real, while the off-diagonal 2×2 block h has elements $\langle\frac{3}{2}, \pm\frac{1}{2}|H|\frac{3}{2}, \pm\frac{3}{2}\rangle$ and $\langle\frac{3}{2}, \pm\frac{3}{2}|H|\frac{3}{2}, \mp\frac{1}{2}\rangle$ representing $\Delta S_1 = 1$ and 2 interactions. Our formal parameter goes on the diagonal $m = \alpha 1$, while h is parameterized linearly by dynamical variables (x, y, p_x, p_y) in $T\mathbb{R}^2_{x,y}$. Associating the interactions with x and y-oscillations, we obtain respective slow-fast resonances 1:1 and 1:(±2). Therefore, the dynamical anologue of the system by Avron et al. [1988, 1989] is a $\mathcal{T}_S$-invariant 1:1:(±2)-resonant Dirac oscillator. Specifically, consider the cubic Hamiltonian

$$H^{1:1:2}_\alpha(\boldsymbol{S}, \boldsymbol{q}, \boldsymbol{p}) = \frac{3}{4\,S^2}\left[H^0_\alpha(\boldsymbol{S}) + H^1(\boldsymbol{S}, x, p_x) + \sqrt{r}\,H^2(\boldsymbol{S}, y, p_y)\right], \tag{3.4a}$$

whose fixed internal parameter $r > 0$ balances the 1:1 and 1:2 resonances. The value of r is important to the internal structure of quantum superbands associated with the semi-quantum eigenvalues λ_b of (3.4a); it does not affect the conical intersection of λ's, the spectral flow, and the Chern numbers $c_2(\Delta_b)$. We focus on the special resonance-matching case of $r = 2$. In the spin-$\frac{3}{2}$ basis (3.3),

$$H^0_\alpha(\boldsymbol{S}) = \alpha\left(\boldsymbol{S}^2 - 3\,S_1^2\right) = -\sqrt{6}\,T^2_0, \tag{3.4b}$$

$$H^1(\boldsymbol{S}, x, p_x) = \frac{\sqrt{3}}{2}\Big(\left[S_1, S_+\right]_+ a_x + \left[S_1, S_-\right]_+ a_x^\dagger\Big), \tag{3.4c}$$

$$\text{and} \quad H^2(\boldsymbol{S}, y, p_y) = \frac{\sqrt{3}}{2}\left(S_+^2\, a_y + S_-^2\, a_y^\dagger\right) \tag{3.4d}$$

define $\Delta S = 0$, 1, and 2 elements[5] of the quaternionic semi-quantum matrix (1.1b)

$$\begin{pmatrix} -\alpha 1 & h \\ h^\dagger & \alpha 1 \end{pmatrix} \quad \text{with} \quad h(\boldsymbol{q}, \boldsymbol{p}) = \begin{pmatrix} \sqrt{r}\, a_y & a_x \\ -a_x^\dagger & \sqrt{r}\, a_y^\dagger \end{pmatrix}. \tag{3.5}$$

For $r = 2$, the multiplicity-2 eigenvalues of this matrix

$$\pm\sqrt{\alpha^2 + I_x + r\, I_y} \xrightarrow{r=2} \pm\sqrt{\alpha^2 + I}$$

depend only on the principal (or polyad) action

$$I = I_x + 2\, I_y \geq 0 \tag{3.6}$$

of slow 1:2-resonant oscillations in (x, y). Since 1-oscillator actions $I_{x,y}$ quantize as $n_{x,y} + \frac{1}{2}$, we quantize I as $n + \frac{3}{2}$, with $n_x, n_y, n \in \mathbb{N}_0$. We notice that, to the new meaning of I and multiplicity, the semi-quantum eigenvalues of the 1:1-resonant (original) Dirac oscillator Hamiltonian (3.2b) and the 1:1:2 spin-oscillator Hamiltonian (3.4) are identical. Furthermore, slow momentum conjugations of (3.4) produce resonances 1:(± 1):(± 2). Of these, 1:1:(-2) with unbound I is obtained through $p_y \mapsto -p_y$ and merits special consideration below.

Quantum spectra of the two-(super)band 1:1 and 1:1:(± 2) systems (fig. 3b) can be easily computed because their Hamiltonians squared are diagonal. So taking the square of Hamiltonian (3.4)

$$\mathsf{H}_\alpha^2\big|_{r=2} = \alpha^2 1 + \operatorname{diag}(\hat{n} + 3,\ \hat{n},\ \hat{n} + 1,\ \hat{n} + 2) \quad \text{with} \quad \hat{n} = \hat{I} - \tfrac{3}{2}\,,$$

we realize that these spectra are essentially the same. Specifically, while all 1:1-levels are nondegenerate, bulk levels of the 1:1:2 system with $r = 2$ have multiplicity $v \in \mathbb{N}$.

It is instructive to consider non-elementary systems with Hamiltonian (3.4) and large half-integer spins $S > \frac{3}{2}$. Their bands $b = 1 \dots S + \frac{1}{2}$ have a complicated isolated degeneracy in 0, which can be deformed into a "constellation" of elementary ones. This degeneracy is characterized by Chern numbers in theorem 6.3 of [Avron et al., 1989, Sadun and Segert, 1989], which we denote $c_2(\Delta_b^S)$ or $c_2\big(\Delta_{|S_1|}^S\big)$ with $|S_1| = \frac{1}{2}, \frac{3}{2}, \dots, S$ and $b = |S_1| + \frac{1}{2}$. We uncover how these numbers replicate the spectral flow. First, we improve the construction in fig. 4b by flipping its negative-S_1 part and fitting all bulk states within a convex wedge domain (of $2 \arctan \frac{1}{2} \sim 53°$, shaded blue in fig. 5a). Explicitly, this can be achieved through height function

$$F(\hat{\boldsymbol{S}}, \boldsymbol{q}, \boldsymbol{p}) = -\sin(\pi S_1) \left(\frac{S_1}{2} + n_y + \frac{1}{2} \right). \tag{3.7}$$

Here we notice that for half-integer S, the front factor

$$\sin(\pi S_1) = (-1)^{|S_1| - \frac{1}{2}} \operatorname{sign}(S_1)$$

takes values ± 1, and for all even b, it flips the lattice additionally about the median. This property of F is displayed by the alternating shade pattern in fig. 5b. Furthermore,

[5] Normally, we do not have to distinguish quantum and classical definitions, but (3.4c) requires an explicit quantum-specific expression with anti-commutator $u\,v + v\,u$ denoted as $[u, v]_+$. Traceless quadratic spin operators in (3.4) are components of spherical tensor $T^2(\boldsymbol{S})$ [Zare, 1988, chap. 5, appendix 13] corresponding to unit spin-quadrupoles $Q_{0\dots 4}$ in [Avron et al., 1989, eqs. (3.1)–(3.3) and proposition 3.6].

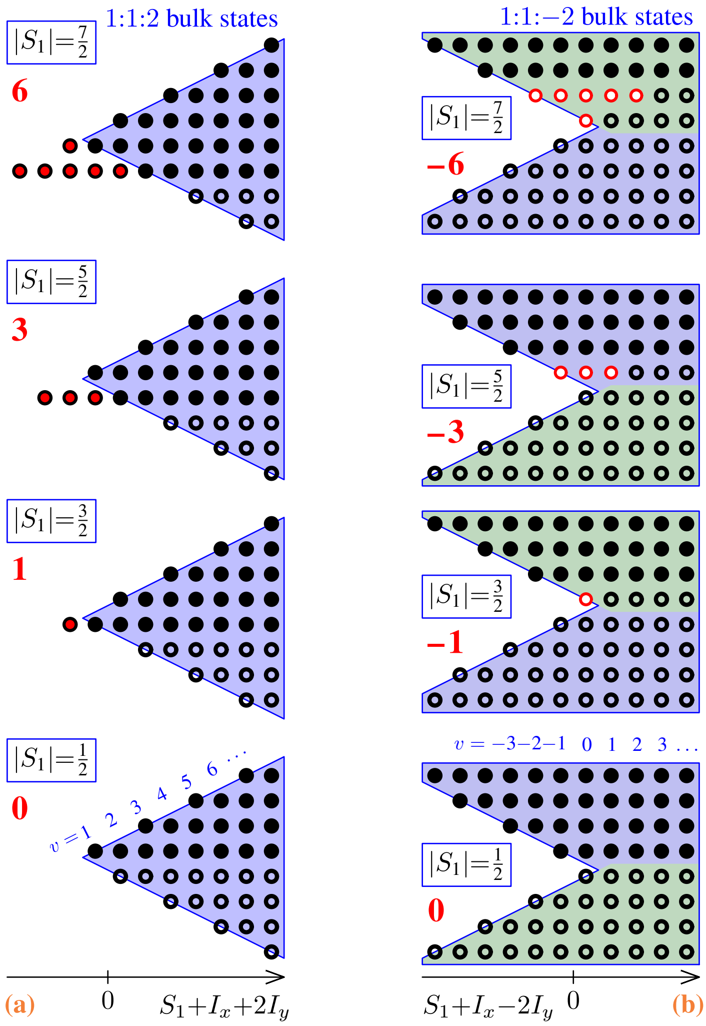

Figure 5: Trivial (uncoupled) bases of the 1:1:2 (a) and 1:1:−2 (b) resonant large-spin Dirac oscillators for $\alpha < 0$. Red solid and opaque circles mark surplus and missing states, whose numbers are indicated in bold large red digits. The vertical axis can be given explicitly by (3.7).

columns (a) and (b) in fig. 5 can be seen as representing S_1-slices of the 3-dimensional lattices of the respective joint eigenspectra of (S_1, J, F). We observe that the structure of superbands depends on $|S_1|$ and does not depend on S, and that all bulk wedges in fig. 5a are identical. The two 1:2-oscillator lattices (fig. 4b) form superbands in such a way that these wedges have straight boundaries. Compared to the $|S_1| = \frac{1}{2}$ superband, larger-$|S_1|$ superbands possess *surplus states* outside the wedges (red dots in fig. 5a). The spectral flow equals the difference of the surplus state numbers (bold red numbers in fig. 5a) in the reciprocal superbands and matches exactly the Chern number

$$c_2\big(\Delta^S_{|S_1|}\big) = v_S\left(\frac{v_S}{2} - |S_1|\right) \quad \text{with} \quad v_S = \frac{2\,S+1}{2}\,,$$

which was computed in [Avron et al., 1989, Sadun and Segert, 1989]. So indices c_2 of superbands $\frac{7}{2} \leftrightarrow \frac{1}{2}$ and $\frac{5}{2} \leftrightarrow \frac{3}{2}$ for $S = \frac{7}{2}$ are read out from fig. 5a as $6-0$ and $3-1$.

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The matching between the spectral flow and Chern indices is beyond any doubt.

Turning to the 1:1:(-2) lattice (fig. 5b), we construct it explicitly using the same height function, and observe its *complementarity* to 1:1:2 after identifying the wedged boundaries on each row of fig. 5 and uniting the two jigsaw puzzle pieces into $\mathbb{Z}^2$. The 1:1:(-2) lattice is wholly unbound, with large-$|S_1|$ slices having missing states (holes). The spectral flow and the c_2 indices are minus those in the 1:1:2-system.

Although the simultaneous crossing of more than two eigenvalues in large-spin elementary $\mathbb{C}$ and $\mathbb{H}$ systems (with $S > \frac{1}{2}$ and half-integer $S > \frac{3}{2}$, respectively) is not generic for the number of available parameters (both formal and dynamical), it can be deformed continuously, or *unfolded* into a sequence of elementary degeneracies involving pairs of neighboring (super)bands and being associated with the transfer of a single quantum state *between the two neighbours*. For the given trivial limit, this can be done in several ways. However, the minimal number $\mathcal{N}_S$ of elementary systems required for the unfolding is well defined and can be rather simply calculated. It suffices to count all states that we need to transfer between neighboring (super)bands in the same energy axis sense in order to arrive at the reciprocal limit. In this way, we find

type	spin	(super)bands	topological charge
$\mathbb{C}$	$S \geq \frac{1}{2}$	$v_S = 2\,S + 1$	$\mathcal{N}_S = v_S\,(v_S^2 - 1)/6$
$\mathbb{H}$	$S \geq \frac{3}{2}$	$v_S = S + \frac{1}{2}$	$\mathcal{N}_S = v_S^2\,(v_S^2 - 1)/12,$

where v_S is the number of (super)bands and $\mathcal{N}_S$ is a topological invariant, which can be regarded as a generalization of the elementary topological charge 1 in sec. 2.

4 Discussion of the results

Our main question (sec. 1) about the dynamical equivalent of the spin-quadrupole system of Avron et al. [1988, 1989] and the dynamical triad completion (fig. 1) received a canonically clear and simple answer. The significance of this result transcends our concrete purpose. We demonstrate that classical, semi-quantum, and quantum limits of slow-fast Hamiltonian dynamical systems can be analyzed in terms of basic elementary $\mathbb{R}$-$\mathbb{C}$-$\mathbb{H}$ forms, whose role is similar to those in the bifurcation theory or the singularity classification in complex geometry. Uncovering the mathematics of the quantum slow-fast singularities, we finalize our understanding of the fundamental physical phenomenon of the energy level redistribution between energy level bands. In the spirit of Simon [1983], we consider Chern numbers $c_k(\Delta)$ describing the Δ-bundles (fig. 2) of semi-quantum eigenstates, and uncover their relation to the spectral flow. The analysis of isolated point degeneracies of semi-quantum (symbolic) systems in their *multi-parameter* space Σ becomes equivalent in the quantum limit to counting solutions (edge states) of corresponding *one-parameter* families of systems of elliptic linear partial differential equations being transferred between bands (of bulk states). The universality of this approach has been apprised in many different fields [Volovik, 2009, Delplace, 2022], see also sec. 3 in [Faure, 2022], where a "topological normal form" for molecules is introduced, and the number of the redistributed energy levels is related to the Chern number. It should be placed next to two other groups of results in the literature: (i) the "folk" theorems "Fredholm index = spectral flow" for rather general families of self-adjoint operators on Hilbert spaces [Atiyah et al., 1976, Robbin and Salamon, 1995], and (ii) the index theorems "Fredholm index = Bott index", see [Atiyah, 1967, 1968], [Higson and Roe, 2008], and chapt. 11 in [Bleecker and Booß-Bavnbek, 2013], or "Fredholm index = Chern number $c_k(\Lambda)$" if the slow phase space P is compact (see [Atiyah and Singer, 1968, Atiyah et al., 1975a,b, 1976] or chapt. 12 in [Bleecker and Booß-Bavnbek, 2013]) and we can describe the spectral flow using topological invariants of individual "bands" or Λ-bundles of semi-quantum eigenstates over P at non-critical values of formal parameter $\alpha \neq 0$, cf. sec. 1.2 of [Iwai et al., 2020].

In the compact setup, the space of formal parameter(s) α is divided by the degeneracy set of the semi-quantum eigenvalues (the slow-fast singularity) into regular iso-Chern domains, each described by its own set of numbers $c_k(\Lambda_b)$, for which $c_k(\Delta_b)$ play the role of "delta-Chern" numbers reflecting changes occurring in $c_k(\Lambda_b)$ when we cross between the domains. The recent quaternionic example on compact $P = \mathbb{S}^2 \times \mathbb{S}^2$ [Sadovskií and Zhilinskií, 2022] generalizes the original spin-orbit system [Pavlov-Verevkin et al., 1988] with $P = \mathbb{S}^2$ and combines all four local elementary slow-fast resonances $1{:}(\pm 1){:}(\pm 2)$, which we describe in this work, into a single one-parameter family of physical systems. Presentation-wise, the quaternionic spin-orbit example with $S = \frac{3}{2}$ in [Sadovskií and Zhilinskií, 2022] should be considered as a sequel to the present work. It describes a more physical model with bounded energies and finite numbers of states in the superbands. The discussion of elementary phenomena and underlying mathematics is inadvertently reduced in favor of the specific features of that model system.

We surveyed several decades of research on quantum systems exhibiting redistribution of states between bands of slow-fast systems. This work advanced along several interconnected directions culminating with the model quaternionic dynamical two-band system. Its results and implications can be grouped in the following way.

1. The concept of elementary (maximally) dynamically parameterized slow-fast singularities, their correspondence to the paradigm systems with geometric phases, and their $\mathbb{R}$-$\mathbb{C}$-$\mathbb{H}$ classification ("Arnold's trinity"), see sec. 2.
2. The specific universal local form of elementary $\mathbb{H}$-systems, see sec. 3.
3. The first analysis of the rearrangement of quantum states in the quaternionic slow-fast dynamical system based on this model.
4. The conjecture and convincing demonstration of the theorem relating this rearrangement and the Chern index of the semi-quantum eigenvalue crossing.
5. The consequences and importance to physics, to classical mechanics, singularity theory, and index theories for partial differential equations.

In conclusion, a theorem is typically a highly nontrivial statement, and one should be well aware of what it is worth before engaging in its formal proofs. We made sure that (i) it is indeed substantial and nontrivial; (ii) there is an informal proof, or a demonstration; (iii) it is of importance to applications in other fields (physics); and we provided ample evidence that (iv) it is important to mathematical theory. There is a large number of mathematical papers discussing "bulk-edge" correspondence across many models associated with topological effects in various physical situations. The physical concept of slow-fast separation is paralleled in the theory of (elliptic) partial differential equations by the fundamental idea of the symbol of the equation. By formulating the relation between the spectral flow and Chern indices in the *physical elementary slow-fast system* as a theorem, we like to bring attention to this important mathematical fact. We anticipate further interest by the experts in index theories, who can, if necessary, develop our statements and provide general proofs.

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