Generalization of Hamiltonian monodromy. Quantum manifestations

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Within the qualitative approach to the study of finite particle quantum systems different possible ways of the generalization of Hamiltonian monodromy are discussed. It is demonstrated how several simple integrable models like non-linearly coupled resonant oscillators, or coupled rotators, lead to physically natural generalizations of the monodromy concept. Fractional monodromy, bidromy, and the monodromy in the case of multi-valued energy-momentum maps are briefly reviewed.

Keywords: monodromy, energy-momentum map, bidromy

1. Introduction

The main goal of the qualitative theory of excited quantum finite particle systems is to describe and to classify generic qualitative phenomena which can be presented in families of Hamiltonian dynamical systems depending on a number of control parameters. The basic tools of such analysis are quantum-classical correspondence, symmetry group actions, topological aspects (for a review see [1–4] and references therein). Initially the accent of the qualitative analysis was put on the "quantum bifurcations" [5–7] and the redistribution of energy levels between bands [8–11] in the energy spectrum under the variation of some control parameters. At the end of the nineties Richard Cushman brings to the attention of physicists the Hamiltonian monodromy phenomenon. The monodromy was known to exist in several simple classical mechanical problems like spherical pendulum [12,13] and was tentatively suggested to be of certain importance for quantum problems [14]. Soon after, the presence of monodromy was demonstrated for a number of different integrable approximations for concrete physical systems like coupled angular momenta [15,16], hydrogen atom in external fields [17–19], H_2^+ molecular ion [20], CO_2 molecule [21,22] and many other simple atomic and molecular systems. As soon as all these real physical systems are quantum, the notion of "quantum monodromy" is needed. It was introduced in [23,24] and the interpretation of quantum monodromy as a certain "defect" of the lattice formed by joint spectrum of several commuting observables for quantum problem was suggested [4,25].

2. Singularities of energy-momentum maps and monodromy

In its simplest form the Hamiltonian monodromy [13,26] appears for classical completely integrable problems with two degrees of freedom [12,27,28]. Such dynamical systems can be considered as integrable toric fibrations defined by two integrals of motion in involution [29]. Typical simplest images of energy-momentum (EM) maps shown in Fig.1 consist of regular values and singular values. The inverse images of regular values are two-dimensional tori (one or several) [30]. Singular values on the boundary of the EM map image correspond to lower dimensional tori. Typical isolated singular values (codimension two singularities) which appear generically for Hamiltonian dynamical systems with two-degrees of freedom are associated with the so called pinched torus (one of the generating circles of the torus is shrinked to a point, Fig.2a) [12,27]. The presence of an isolated singular fiber makes toric fibration non-trivial. The monodromy describes the global twisting of the family of tori parameterized by a closed path going through regular values of the EM map of the integrable system. It can be considered as an automorphism of the first homology group of regular fibers associated with the homotopy equivalent class of closed loops on the regular part of the image of EM map. From the dynamical system point of view the monodromy is the first obstruction to the existence of global action-angle variables for completely integrable problems [13,26].

Pinched torus singularity is structurally stable under small perturbations which preserve the integrability of the problem. That is why the presence of monodromy is important from the point of view of physical applications. Moreover, having a topological origin, the monodromy should in some sense persist even under small non-integrable perturbation. This fact was recently proven for nearly integrable system in the style of KAM theorem [31].

For a family of integrable systems depending on parameters, the position of a singular value on the image of EM map can change and, in particular, this singular value can touch the boundary of the EM map (compare sub-figures (a) and (b) in Fig.1). The corresponding qualitative modifica-

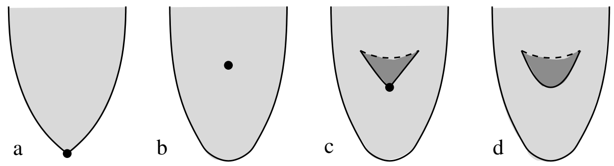

Fig. 1. Typical images of the energy momentum map for completely integrable Hamiltonian systems with two degree of freedom in the case of: (a) - all internal points are regular, no monodromy; (b) - isolated singular value with integer monodromy, (c) - "island" formed by second component with nonlocal monodromy, and (d) - "island" formed by second component with trivial monodromy. [Dashed line - "bitorus" singularity.]

tion of the image of EM map (often named as bifurcation diagram [29]) is the Hamiltonian Hopf bifurcation [18,32]. Another possible qualitative modification which can happen with isolated singular value is another kind of bifurcation which leads to formation of second connected component (or a second fiber). Two components fuse together at a singular line (see Fig.1c,d). Each regular point at that line has a singular "bitorus" as inverse image (see Fig.2b). Note, that the Hamiltonian Hopf bifurcation leads to the formation of an island (second leaf) on the image of the EM map which can be surrounded by a closed loop possessing the same monodromy as the initial singular (pinched) torus (compare sub-figures (b) and (c) in Fig.1). The corresponding quantum problem possesses two different lattices in the two-component region which fuse together along the singular line. Such structure appears in quadratic spherical pendulum [18], LiCN, HCN molecules [33,34], hydrogen atom in external fields [19,35]. At the same time, it should be noted that the "island" formed by the second component can be formed without an initial singular pinched torus. In such a case there are only two exceptional singular values on the singular boundary of "island" associated with the two ends of the "bitorus" line. The corresponding fibers are singular tori shown in Fig.2c. The monodromy associated with the closed loop around such an island is trivial (identity).

The standard definition of Hamiltonian monodromy requires the existence of a closed loop in the plane of values of integrals which goes only through regular values of the EM map. An important generalization of this notion was recently proposed which allows the existence of some one-dimensional singular strata which can be crossed by a loop. The corresponding singular fibers are curled tori (Fig.2d). The restriction imposed by the existence of such strata leads to the possibility to define monodromy only for certain subgroups of the first homology group of regular fibers because

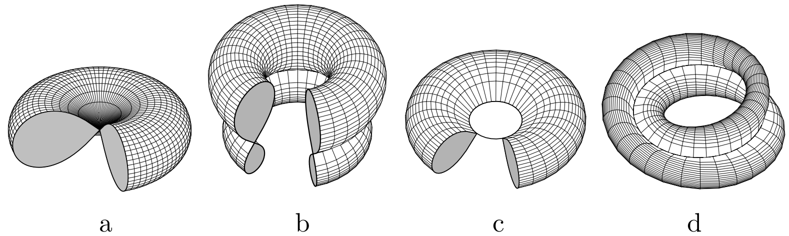

Fig. 2. Two dimensional singular fibers in the case of integrable Hamiltonian systems with two degrees of freedom (left to right): a - pinched torus, b -bitorus, c - singular torus, and d - curled torus.

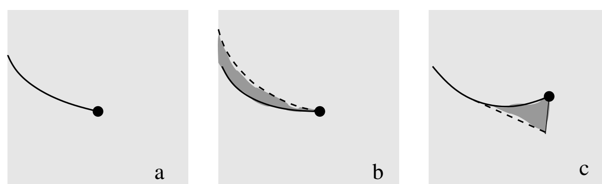

Fig. 3. Typical images of the energy momentum map for completely integrable Hamiltonian systems with two degree of freedom in the case of fractional monodromy (black point correspond to essential singularity): (a) - singular line is formed by curled tori; (b) - "island" formed by second connected component with one of its boundary being a circle with nontrivial stabilizer (fuzzy fractional monodromy [40]); (c) - "island" formed by second component (non-local fractional monodromy [35]).

only a subgroup of cycles can go through the singular stratum. The resulting fractional monodromy was first introduced in [36,37] for a problem of nonlinearly coupled resonant oscillators and was illustrated immediately on quantum example by the evolution of a multiple (double) cell along a closed path crossing once the singular stratum. Much more detailed analysis of the fractional monodromy is given in several recent publications [37–40].

In its simplest version, the fractional monodromy is defined for closed paths which do not cross singular strata of the bitorus type, i.e. singular fibers associated with transformation of one regular torus into two regular tori. At the same time, similar to the case of integer monodromy, the pinched curled torus, i.e. the fiber corresponding to singular value at the end of "curled torus line", can be deformed into an island formed by a second connected component. In such a case, the nonlocal fractional monodromy which is illustrated in Fig.3c arises. In this case the essential singularity, namely the pinched curled torus, gives the second component attached to the main leaf through the bitorus line (see an example of the formation of such structure in the case of hydrogen atom in external fields in [35].) Another possibility of formation of second component is shown in Fig.3b. Here the essential singularity remains on the main leaf, and the second component is attached to the main leaf through the bitorus line. This situation was the subject of Nekhoroshev's study of arbitrary fractional monodromy [40]. He has shown that locally, close to essential singularity, the notion of fractional monodromy can be conserved even if the closed path crosses the bitorus line where the discontinuity of actions takes place. The presence of these discontinuities results in the appearance of "fuzzy" fractional monodromy. The fuzziness becomes less pronounced when the crossing point between closed loop and the bitorus line approaches an essential singularity.

3. Bidromy

Another possibility to cross the bitorus line and to go from the region of EM map with one connected component as inverse image into the region with two connected components was suggested by Sadovskii and Zhilinskii [22,41] on the basis of their study of a realistic physical model describing three-dimensional nonlinear oscillators in the presence of axial symmetry with 1:1:2 resonance and small detuning between a doubly degenerate mode and a non-degenerate one, which corresponds e.g. to the Fermi resonance in the CO_2 molecule. In this problem the two-components and the bitorus line are again present in some region of the EM map image, but their arrangement now is completely different (see Fig.4). In fact, the two components are formed due to self-overlapping of one regular leaf of the EM map. As Fig.4 schematically shows, the two fibers (b', b'') associated with the same values of the integrals are two different regular tori, which can be deformed one into another along a path going entirely through regular fibers. Such path starts and ends at different components and, consequently, it is not closed even if its initial and final points have the same values of the integrals. In such situation the interesting possibility of defining a "bipath" appears [22].

Let the "bipath" start at point a (Fig.4) and go through regular tori till point c at the bitorus line. At that point the path ac bifurcates into two component path. Each sub-path $[(cb')$ and $(cb'')]$ evolves through regular tori on different components in the region of self-overlapping. The two sub-paths of the bipath go independently through regular tori till the initial point where they fuse. The only singularity on the bipath is the bitorus c. Point a is special, but the corresponding fiber is regular.

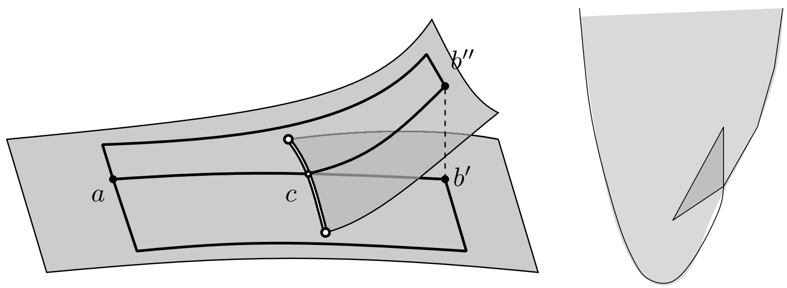

Fig. 4. Schematic representation of a bipath associated with bidromy.

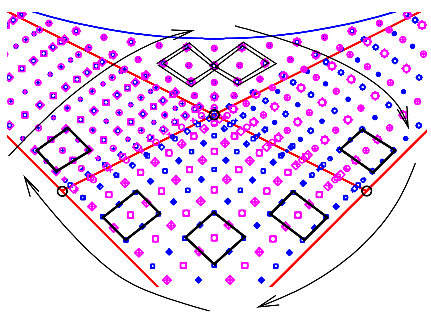

Fig. 5. Energy momentum diagram for the Manakov top together with joint spectrum of mutually commuting operators for the corresponding quantum problem (taken from [44]). Inverse images of regular points in four regions consist of two, two, two and four regular tori. Evolution of a quadruple cell along a closed path surrounding the central singularity consists in splitting of quadruple cell into two double cells when entering into region with four components. The transformation between the initial and the final cell is trivial.

The following transformation of quantum cells of the joint spectrum lattice is suggested to be associated with such a bipath [22]. We start at a with the double cell which splits into two cells belonging to the different components when crossing the bitorus line. Further evolution of each cell is regular till the final point where two cells should fuse together. The resulting cell can be compared with the initial one. The transformation between initial and final cells is named the bidromy transformation. It is conjectured that this transformation does not depend on the place where the bipath crosses the bitorus line. It is quite important to note that in the case of the "island" monodromy similar crossing of the bitorus line leads to a result which apparently depends on the position of the crossing point.

The Manakov top problem [42–44] gives an another interesting example of the EM map with an even more complicated system of connected components (see Fig.5). The recent paper [44] uses a construction similar to "bipath" and bidromy, which allows to define the evolution of a multiple quantum cell along a "multicomponent" path. Curiously, the generalized monodromy defined in this case becomes trivial, in spite of using rather complicated multi-paths.

4. Conclusions

We briefly reviewed the possible generalizations of the monodromy concept which lead to fractional monodromy and bidromy. While presently the mathematically rigorous definition of fractional monodromy seems to be properly formulated, the description of "bidromy" is just at its initial stage. Further mathematical constructions are necessary, and quite demanding, in order to propose mathematically satisfactory tools which allow to treat the new qualitative features in dynamics which were heuristically described on several examples of quantum molecular models.

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