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Avant-Propos

Symmetry, invariants, topology

Nearly seventy years have elapsed since the publication of the two great classic books by Weyl (1931) and by Wigner (1959), dealing with group theory and its application to quantum mechanics. More than a hundred books have appeared on the same subject; they are also centered on linear representations of groups. In this Volume, the emphasis is on group actions and their decomposition into orbits and strata; the study of the corresponding orbit space and of the set of strata is basic in physics. Linear representations are only a particular case of group actions; their decomposition into strata, made under different names in the different domains of physics, is here unified. Except for some part of Chapter III dealing with atoms, the field of applications is restricted to molecules and crystals whose symmetry groups are essentially discrete. This seems to make paradoxal the appearance of the word topology in the title, but the source of the topological problems comes from the topology of the space on which the symmetry group G acts: phase space, surface of constant energy, Brillouin zone…. The topological consequences are very varied and often new in physics. They give a "qualitative" study of phenomena. The qualitative approach is reviewed mainly for molecules in Chapters II and III and for crystals in IV (Section 6), V and VI.

Chapter I is a necessary introduction to topics not found in the hundreds of books we mentioned. Probably many readers, after a first reading which will acquaint them with the presented background, will use it more as a dictionary to be consulted if needed for the study of the other Chapters or in their future works! In Chapter I, group actions are introduced and their properties, which will be used later, are explained; for their proofs we refer to the original literature but diverse examples of group actions are studied, most of them to be used later. The action of G on a space M is naturally transferred to the action of G on the functions defined on M. In all the applications presented in this issue, there exist "critical orbits" which are the orbits of extrema for *all* invariant functions (Michel, 1971) and, for each function, the existence of other orbits of extrema on some strata can be predicted (Michel, 1971). For G-invariant Morse functions, the predictions of Morse theory in the presence of symmetry are explained; they require the existence of a minimum number of extrema of different nature (minima, or maxima, or saddle points). Important examples are given in the different chapters.

In Chapter I, we also study the rings of invariant functions and more specially those of invariant polynomials with several examples. This study also allows a "qualitative" approach of many phenomena; a simple method is the study of level functions on orbit spaces (it has been introduced by Kim (1982, 1984) in high-energy physics and by Zhilinskii (1989a, 1989b) for molecules). Some work, for some space groups, had been made since forty years on types of invariant functions on the Brillouin zone. In Chapter V, the problem is completely solved for all space groups; the results are

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given as modules of invariant polynomials and presented in a dozen of short tables. It is remarkable that most of these modules have so few generators.

In Chapter VI, the same philosophy is applied to the symmetry and topology of electron energy bands in crystals. The basic hypotheses have been made by Zak (1980, 1982a, 1982b). It has been possible to deduce their full consequences only in the last four years; this is a general presentation of the results.

It must be emphasized that this monograph is not written for specialists; for instance, to help readers who have never worked in solid state physics, Chapter IV is an introduction to crystallography (using concepts defined in Chapter I). It has also to be read by solid state physicists because it emphasizes basic concepts (e.g. arithmetic class) and tools (necessary for the next two chapters) which are not even mentioned in usual text books. Moreover, some of the sections (e.g. 6 and 7) of this chapter contain completely new material.

This monograph is written for readers who want to learn methods, either new or not enough common, for studying symmetry in physics and the "qualitative" or topological approach of the rigorous consequence of symmetries. We hope that they will enjoy the applications to molecular and crystal physics presented here. But *the main aim of the authors of this monograph is to suggest to the reader to apply these tools to other domains of physics!* For this, we sometimes include (generally in smaller print or in appendix) glimpses of theories not strictly necessary for the studied examples but which may be necessary for other applications.

Seeing analogy between some well understood and some new puzzling phenomena is a natural way of discovery in science; and it is very fruitful when the two classes of phenomena can be described by the same type of mathematics. For a more advanced field of science, "qualitative" mathematical approach is a way of "thinking" about phenomena for the next generation of scientists. We see a deep truth in the sentence written by Galilei (1623) at the dawn of modern science:

> "La filosofia è scritta in questo grandissimo libro, che continuamente ci sta aperto innanzi agli occhi (io dico l'universo), ma non si può intendere se prima non s'impara a intender la lingua, a conoscer i caratteri, ne' quali è scritto. Egli è scritto in lingua mathematica ..."[1]

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[1] A summary in English: "The great book of the universe stays open before our eyes; but to understand it, we have first to learn the language in which it is written, the mathematics."

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