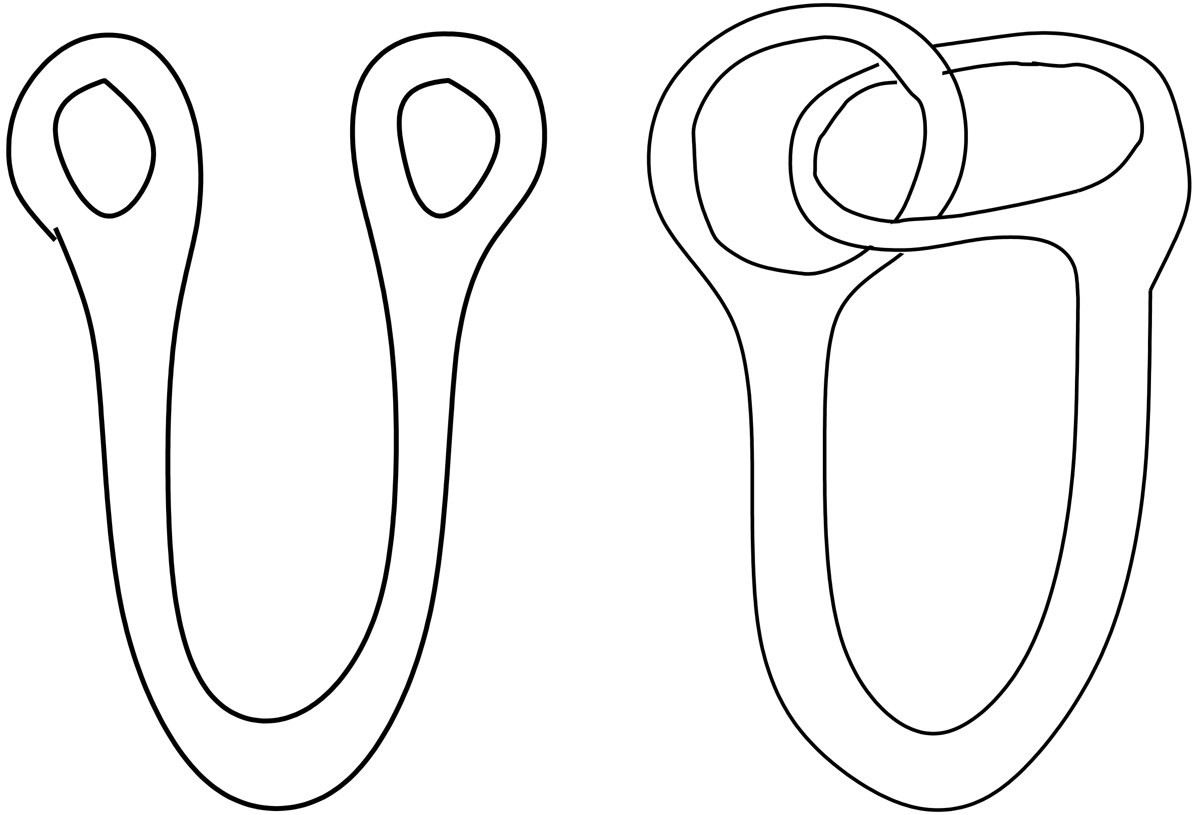
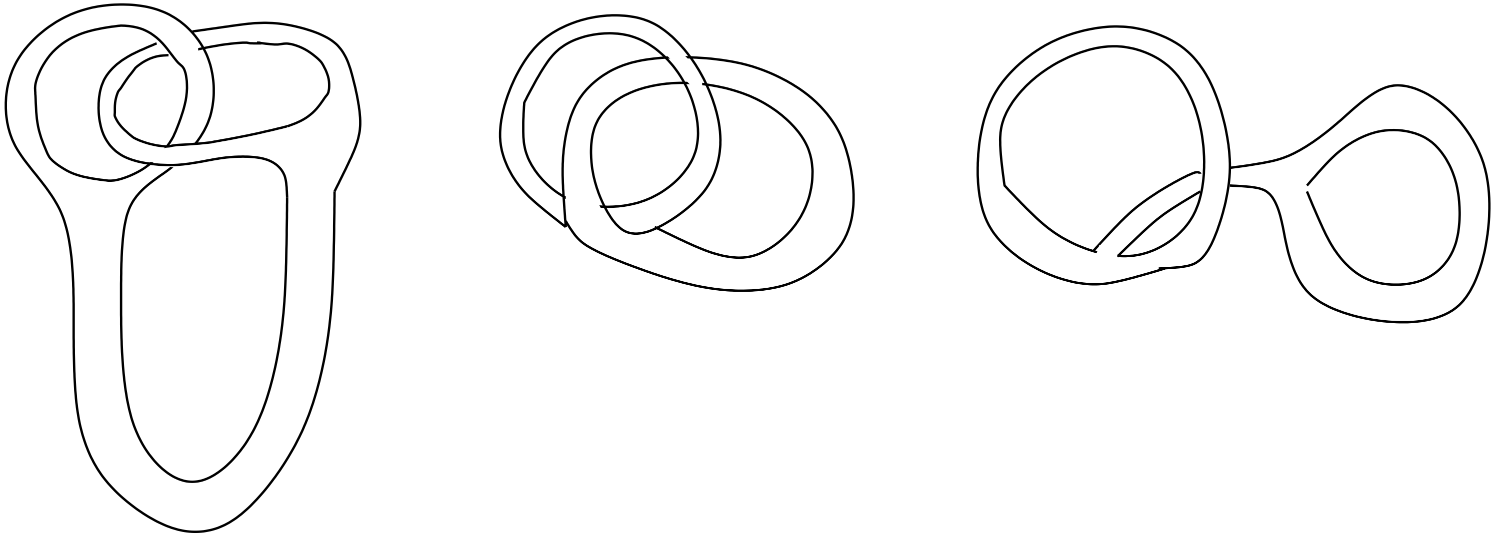
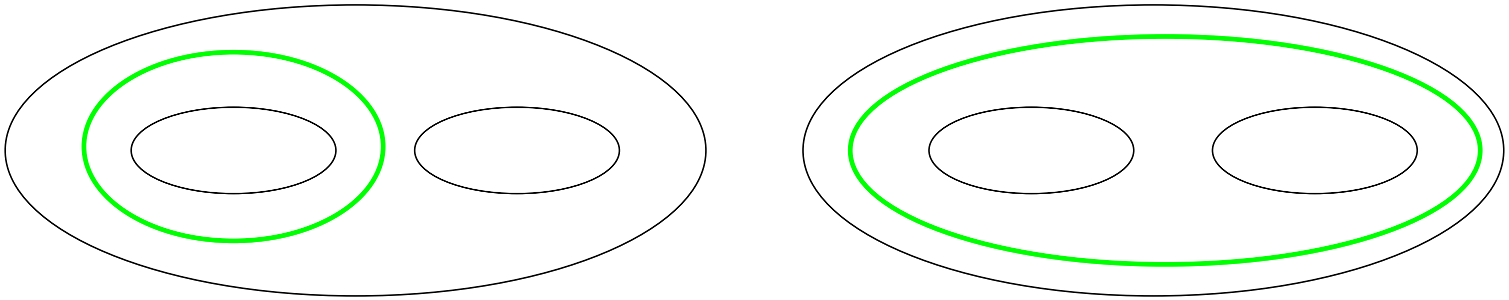
# TOPOLOGY

1. Deformation of elastic body.
2. Cutting and gluing.
3. Morse type functions.
4. Morse inequalities.
5. Morse functions and symmetry.
6. Generating functions for Morse polynomials.



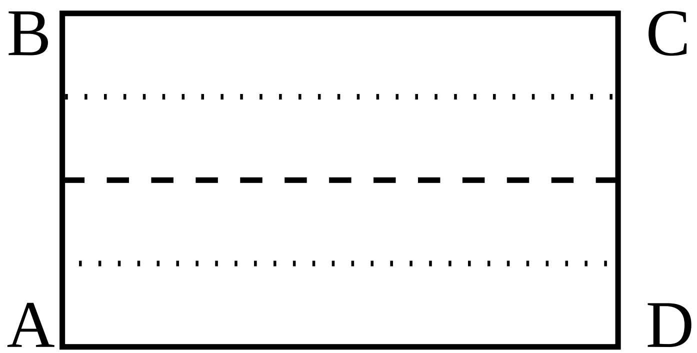
Are these objects topologically equivalent?

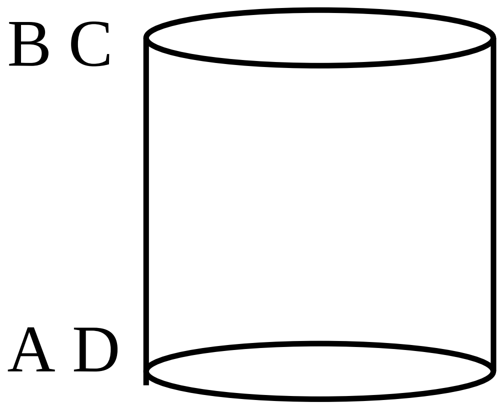




Can one object be transformed into another one?

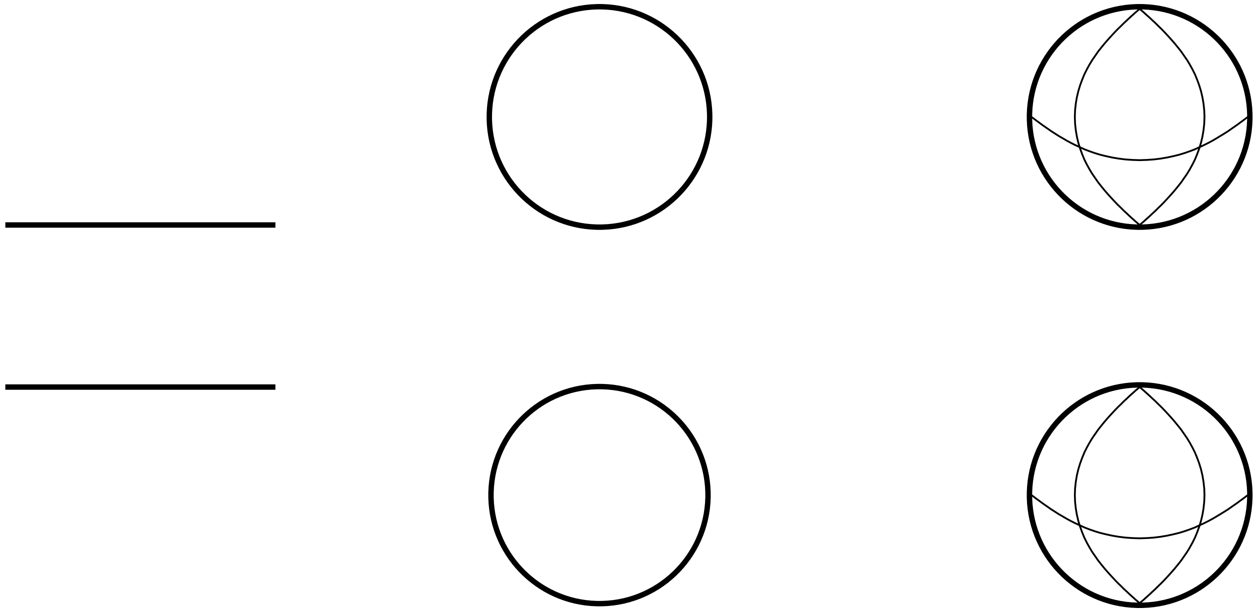
Compare cylinder and Möbius strip.



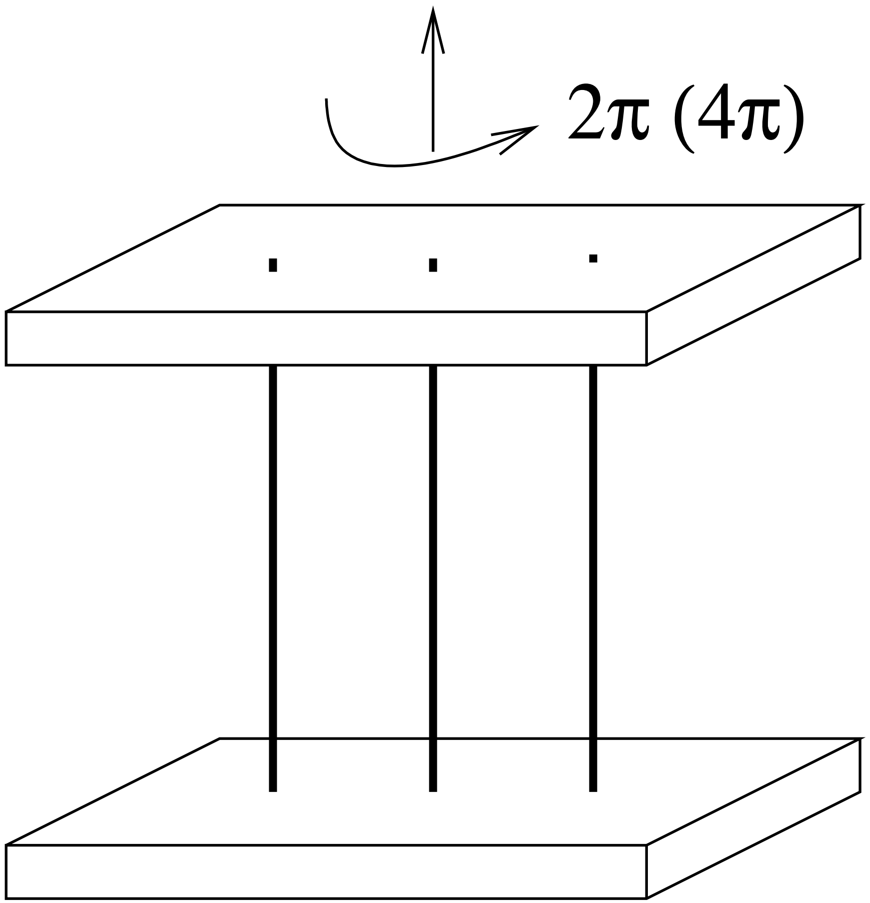
Gluing handles to sphere.

Gluing disks.



$D_1$ $D_2$ $D_3$

Dirac construction which demonstrate non-equivalence of $2\pi$ and $4\pi$ rotations.



# Morse functions

Function $F(x_1, x_2, \ldots, x_k)$ is a Morse type function if all its stationary points are non-degenerate.

By an appropriate change of variables near each stationary point the Morse type function can be written as

$$-y_1^2 - y_2^2 - \ldots - y_l^2 + y_{l+1}^2 + \ldots + y_k^2$$

The number of plus or minus signs characterize the type of stationary point.

Function of *one variable* defined on the *circle.*
Number of minimum points equals the number of maximum points.

*Compare with Euler relation for polygons.*

For function of *two variables* on the *sphere* the number of maximum points plus the number of minimum points minus number of saddle points is equal to 2.

*Compare with Euler relation for polyhedrons.*

Betti numbers.

$b_0$ - number of connected components.

$b_1$ - number of non-contractible closed curves (circles) which cannot be expressed as a combination of others.

$b_2$ - number of closed non-equivalent 2D-surfaces ...

Table 1: Betti numbers and Euler characteristics for the infinite plane, the sphere and the torus,

| | $b_0$ | $b_1$ | $b_2$ | $\chi = b_0 - b_1 + b_2$ |
|---|---|---|---|---|
| plane | 1 | 0 | 0 | 1 |
| sphere | 1 | 0 | 1 | 2 |
| torus | 1 | 2 | 1 | 0 |

# Morse inequalities.

For a function of two variables defined on 2D-space characterized by Betti numbers $b_0, b_1, b_2$ there exist Morse inequalities

$$c_0 \geq b_0, \tag{1}$$

$$c_1 - c_0 \geq b_1 - b_0, \tag{2}$$

$$c_2 - c_1 + c_0 = b_2 - b_1 + b_0. \tag{3}$$

which include the equality for Euler characteristics. Here $c_i$ is the number of stationary points of index $i$.

What is the minimal number of stationary points for a generic Morse type function defined on the sphere $S_2$ in the presence of symmetry

$C_2$, $D_2$, $T_d$, $O_h$, $I_h$ ?

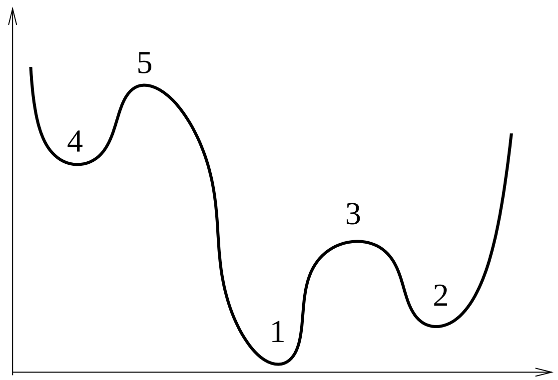
For an island with mountains one calculates:
the number of peaks (max) plus the number of wells (min) minus the number of saddle points.
What is the result?

# Morse polynomials and up-down permutations

Let us consider Morse polynomials of one variable $x$ of degree $n+1$ with the coefficient at $x^{n+1}$ equal 1.

$$p(x) = x^{n+1} + a_1 x^n + a_2 x^{n-1} + \ldots + a_{n+1}$$

We can associate each Morse polynomial with certain permutation on the set of $n$ elements. The permutation indicates the order of critical values of polynomial.



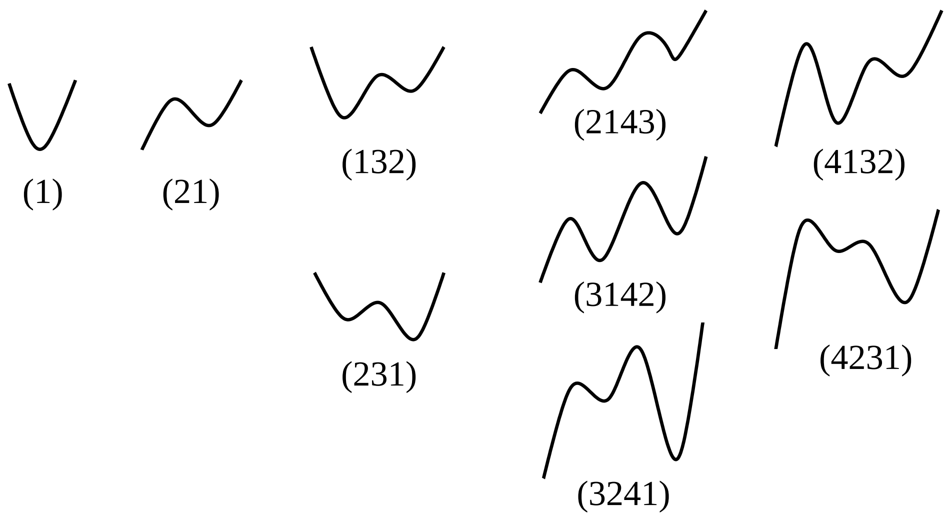

(4 5 1 3 2)

## Up-down permutations

Each element of such permutation either larger than both its (left and right) neighbors or smaller than both neighbors. The last element should always be smaller that its left neighbor.

For small $n$ the numbers of types of Morse polynomials form sequence

$$1, 1, 2, 5, 16, 61, 272, \ldots$$



# DEVIATION: Exponential generating functions

$$\{a_n\} \mapsto \sum_{n=0}^{\infty} \frac{a_n s^n}{n!}$$

Properties of exponential generating functions

Formula for product

$$\left(\frac{a_0}{0!} + \frac{a_1}{1!}s + \frac{a_2}{2!}s^2 + \dots\right)\left(\frac{b_0}{0!} + \frac{b_1}{1!}s + \frac{b_2}{2!}s^2 + \dots\right) = \left(\frac{c_0}{0!} + \frac{c_1}{1!}s + \frac{c_2}{2!}s^2 + \dots\right)$$

where

$$c_n = \binom{n}{0} a_0 b_n + \binom{n}{1} a_1 b_{n-1} + \dots + \binom{n}{n} a_n b_0$$

Derivative and integral of exponential generating functions

$$\left(\frac{a_0}{0!} + \frac{a_1}{1!}s + \frac{a_2}{2!}s^2 + \ldots\right)' = \frac{a_1}{0!} + \frac{a_2}{1!}s + \frac{a_3}{2!}s^2 + \ldots;$$

$$\int ds \left(\frac{a_0}{0!} + \frac{a_1}{1!}s + \frac{a_2}{2!}s^2 + \ldots\right) = \frac{a_0}{1!}s + \frac{a_1}{2!}s^2 + \frac{a_2}{3!}s^3 + \ldots.$$

… end of deviation …

For odd $n$ the number of up-down permutations, $b_n$, is given by the exponential generating function

$$\mathcal{B}(x) = \frac{b_1}{1!}x + \frac{b_3}{3!}x^3 + \ldots = \frac{1}{1!}x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 + \ldots = \tan x$$

For even $n$ the number of up-down permutations, $e_n$, is given by the exponential generating function

$$\mathcal{E}(y) = 1 + \frac{e_2}{2!}y^2 + \frac{e_4}{4!}y^4 + \ldots = 1 + \frac{1}{2!}y^2 + \frac{5}{4!}x^4 + \frac{61}{6!}y^6 + \ldots = \frac{1}{\cos y}$$

## Derivation of the generating function

For each Morse polynomials we move the highest maximum into infinity. This gives the correspondence between initial Morse polynomial and two new Morse polynomials. If initial polynomial has $(n+1)$ critical points, two new polynomials have $k$ and $(n-k)$ critical points with $(n-k)$ being odd.

If $(n+1)$ is odd then $k$ and $(n-k)$ are both odd and we have a recurrent relation for $b_n$:

$$b_{n+1} = \sum_{k \text{ odd}} \binom{n}{k} b_k b_{n-k}.$$

Using properties of exponential generating functions we have

$$\mathcal{B}'(x) = \mathcal{B}^2(x) + 1.$$

Solution of this equation gives

$$d\mathcal{B} = (\mathcal{B}^2 + 1)dx, \quad \int \frac{d\mathcal{B}}{\mathcal{B}^2 + 1} = \int dx$$

$$\arctan \mathcal{B} = x, \quad \mathcal{B}(x) = \tan(x).$$

If $(n+1)$ is even then $k$ is even and $(n-k)$ is odd. The recurrent relation takes the form:

$$e_{n+1} = \sum_{k \text{ even}} \binom{n}{k} e_k b_{n-k}.$$

It corresponds to the following equation on generating functions:

$$\mathcal{E}'(y) = \mathcal{E}(y)\mathcal{B}(y).$$

Its solution gives

$$\frac{\mathcal{E}'(y)}{\mathcal{E}(y)} = \mathcal{B}(y), \qquad (\ln \mathcal{E}(y))' = \tan y,$$

$$\ln \mathcal{E}(y) = \int \tan y \, dy, \qquad \mathcal{E}(y) = \frac{1}{\cos y},$$

# Dynamical system applications

Examples of classifications of dynamical systems.
Definition domain of dynamic variables - phase space topology.

Harmonic oscillator, $\{q_i, p_i\}$ dynamic variables: $R^{2N}$ phase space.

Reduced isotropic harmonic oscillator. $\sum_i (p_i^2 + q_i^2)/2 = Const$
$CP_{N-1}$ phase space.

Rotator with fixed square of the angular momentum.
Dynamic variables $L_x.L_y, L_z$, $L_x^2 + L_y^2 + L_z^2 = const$.
The phase space is a two-dimensional sphere, $S^2 \sim CP_1$.

Two dynamical systems are equivalent if :

o) Their phase spaces are topologically equivalent (very weak equivalence).

i) Their systems of stationary points of Hamiltonians are equivalent.

ii) Stratifications of the phase space by symmetry group action are equivalent.

iii) Their phase portraits are equivalents (very strong equivalence).

Rotational problem ($S^2$ phase space) is equivalent to reduced vibrational problem for two-dimensional harmonic oscillator ($CP_1$ phase space)

- topological equivalence of phase spaces.

Action of the same symmetry group on rotational and vibrational variables can be different.

## Complexity characterization of Hamiltonians.

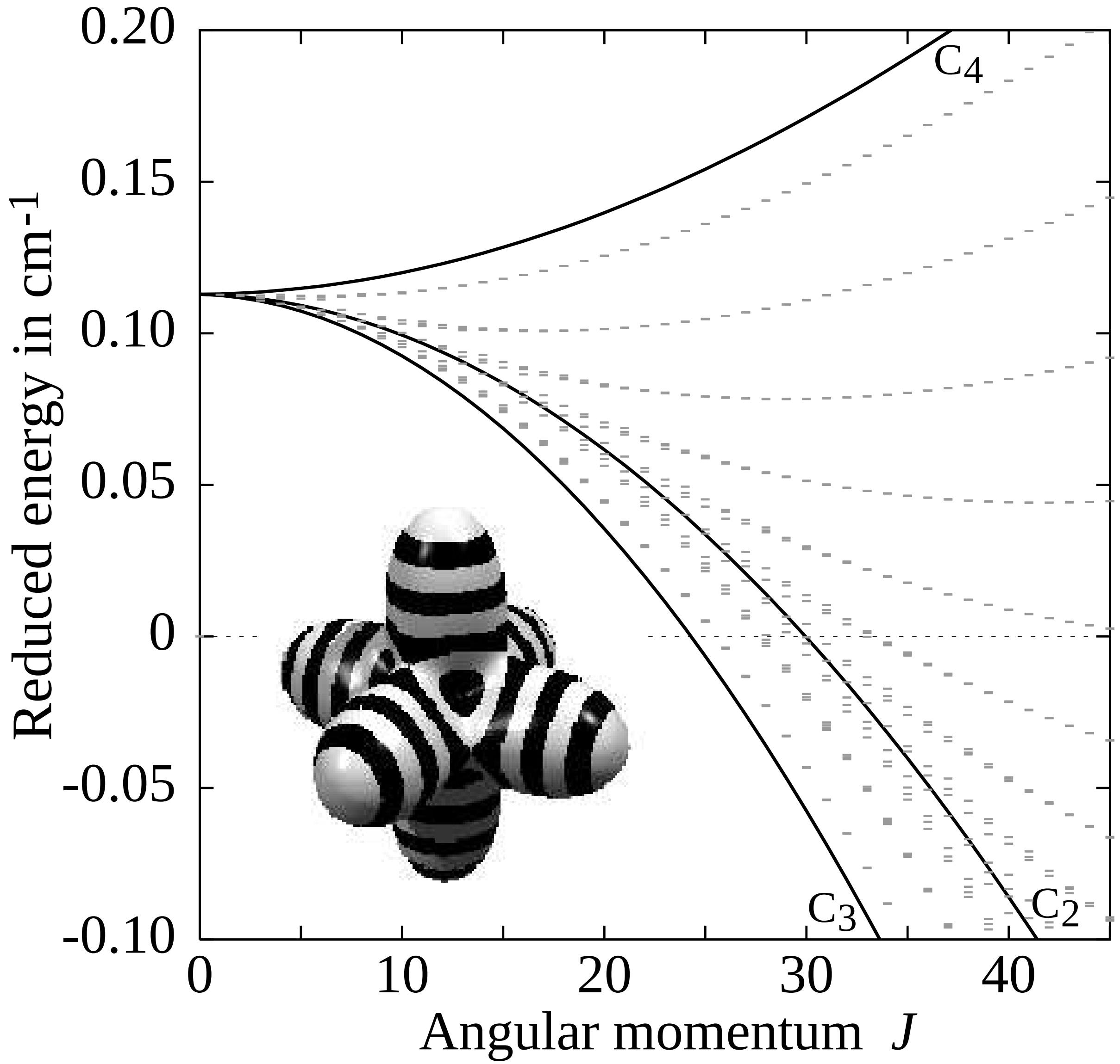
Hamiltonian is a Morse type function.

Simplest Hamiltonian - minimal number of non-degenerate stationary points (compatible with the symmetry group action).
- Perfect Morse type function.

### Examples

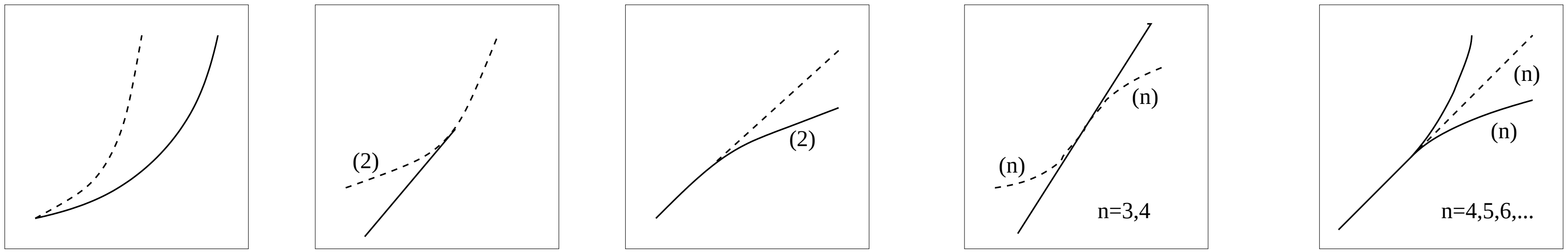
Hamiltonian on the $S^2$ phase space without symmetry.
Two stationary points (minimum and maximum)

Hamiltonian on the $S^2$ phase space in the presence of inversion symmetry.
Six stationary points (two minima, two maxima, two saddles).


0.20
0.15
0.10
0.05
0
-0.05
-0.10
0
10
20
30
40
Reduced energy in cm-1
Angular momentum J
C4
C3
C2

*1 degree of freedom; 1 control parameter*

Table 2: Bifurcations in the presence of symmetry. Solid lines denote stable stationary points. Dash lines - unstable stationary points. Numbers in parenthesis indicate the multiplicity of stationary points.




$C_1^{\pm}$ $\quad C_2^{N\pm}$ $\quad C_2^{L\pm}$ $\quad C_n^N,\ n = 3, 4$ $\quad C_n^{L\pm},\ n \geq 4$

The type of bifurcation depends on the local symmetry (stabilizer) of a stationary point.

Let us consider the transformation of vectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ given by matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{aligned}
A\mathbf{e}_1 &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad ; \quad A^2\mathbf{e}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \\
A^3\mathbf{e}_1 &= \begin{pmatrix} 13 \\ 8 \end{pmatrix} \quad ; \quad A^4\mathbf{e}_1 = \begin{pmatrix} 34 \\ 21 \end{pmatrix} \\
A\mathbf{e}_2 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ; \quad A^2\mathbf{e}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \\
A^3\mathbf{e}_2 &= \begin{pmatrix} 8 \\ 5 \end{pmatrix} \quad ; \quad A^4\mathbf{e}_2 = \begin{pmatrix} 21 \\ 13 \end{pmatrix}
\end{aligned}$$

The integer numbers which appear belong to the Fibonacci sequence.

Matrix $A$ is symmetric, it can be diagonalized.

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1,$$

This gives : $\lambda_+ = \frac{3+\sqrt{5}}{2} = 1 + \varphi$ and $\lambda_- = \frac{3-\sqrt{5}}{2} = 2 - \varphi$.
$\varphi$ is the gold number, solution of equation $x^2 = x + 1$.

Eigenvectors of $A$ :

$$(1+\varphi) \Longrightarrow \begin{pmatrix} 1-\varphi & 1 \\ 1 & -\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2+\varphi}} \begin{pmatrix} \varphi \\ 1 \end{pmatrix}$$

$$(2-\varphi) \quad \Longrightarrow \begin{pmatrix} \varphi & 1 \\ 1 & -1+\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\frac{1}{\sqrt{2+\varphi}} \begin{pmatrix} -1 \\ \varphi \end{pmatrix}$$

$$\text{Matrix} \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

can be considered as a transformation of a torus.

In this context it is known under the name : Arnold‘s cat map

# Catastrophe theory

We want to study the behavior of a system under variation of some control parameters.

The number of parameters is of crucial importance.

Important notions:

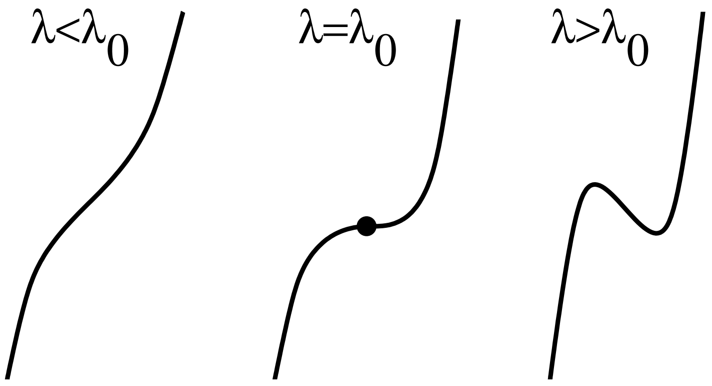
*Generic property*

*Structural stability*

*Examples of generic statements:*

- Generic smooth function of one variable $F(x)$ takes value 0 at some point $x = x_0$ and at that point the first derivative is non-zero, $\frac{dF}{dx}|_{x=x_0} \neq 0$.
- Generic smooth function of several variables $F(x_1, ...x_k)$ is of Morse type (all stationary points are non-degenerate).
- A one-parameter family of smooth functions of one variable $F(x;\lambda)$ has at some value of control parameter $\lambda = \lambda_0$ function $F(x;\lambda_0)$ possessing at some value $x = x_0$ a degenerate stationary point with

$$\frac{dF(x;\lambda_0)}{dx}\Big|_{x=x_0} = 0 \quad \text{and} \quad \frac{d^2F(x;\lambda_0)}{dx^2}\Big|_{x=x_0} = 0$$

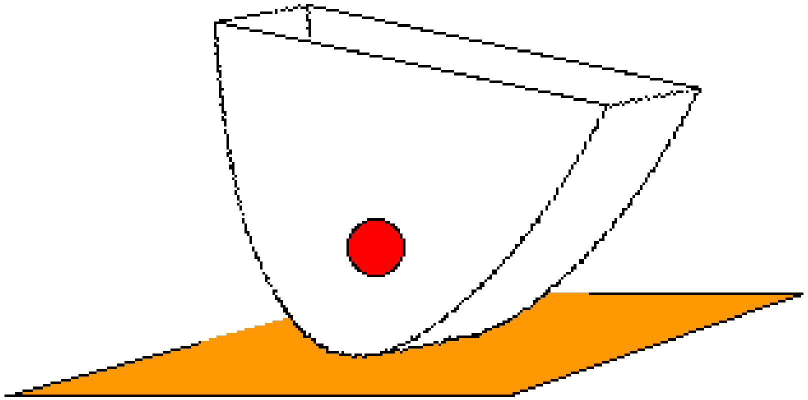


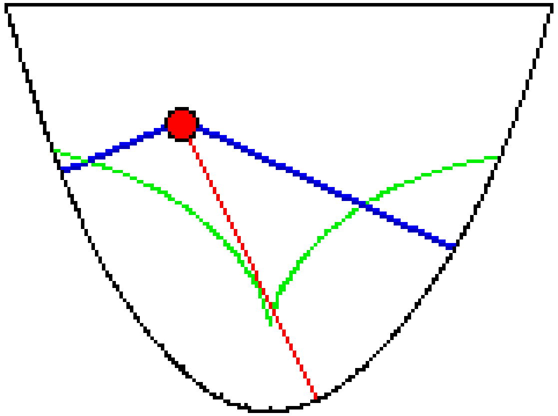

*Fold catastrophe.*

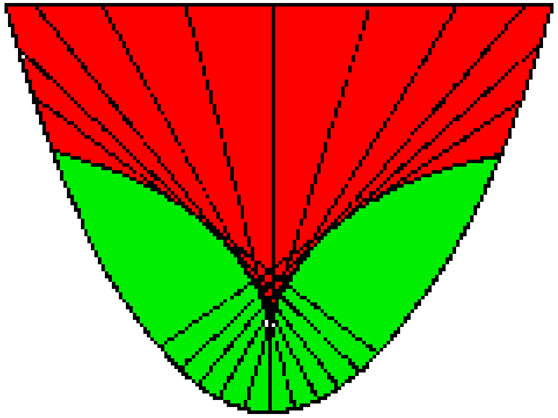
- Eigenvalues of a generic real symmetric matrix $M$ are not degenerate.
- One parameter family of real symmetric matrices $M(\lambda)$ does not include matrices with degenerate eigenvalues.
- Two-parameter family of complex Hermitian matrices $M(\lambda, \mu)$ does not include matrices with degenerate eigenvalues.
- Three parameter family of complex Hermitian matrices $M(\lambda.\mu, \nu)$ has for some isolated values of parameters $\lambda = \lambda_0, \mu = \mu_0, \nu = \nu_0$ matrix $M(\lambda_0, \mu_0, \nu_0)$ with two degenerate eigenvalues.

Evolution of the equilibrium position as a function of control parameters.

# Gravitational machine







# Van der Waals equation

$$(P + \frac{a}{V^2})(V - b) = RT$$

For $T > T_{\rm cr}$ - $P(V)$ has no stationary points.
For $T < T_{\rm cr}$ - $P(V)$ has one minimum and one maximum.

[The line of the first order phase transition ends by a critical point.]

$$PV^3 - bPV^2 + aV - ab = RTV^2 \quad (4)$$

$$V^3 - (b + \frac{RT}{P})V^2 + \frac{a}{P}V - \frac{ab}{P} = 0 \quad (5)$$

For critical isoterm $T = T_{\rm cr}$ this polynomial has three coinciding roots

$$(V - V_{\rm cr})^3 = V^3 - 3V_{\rm cr}V^2 + 3V_{\rm cr}^2 V - V_{\rm cr}^3 = 0 \quad (6)$$

Compairing coefficients allows to find $V_{\rm cr}, P_{\rm cr}, T_{\rm cr}$

$$3V_{\rm cr} = b + \frac{RT_{\rm cr}}{P_{\rm cr}} \tag{7}$$

$$3V_{\rm cr}^2 = \frac{a}{P_{\rm cr}} \tag{8}$$

$$V_{\rm cr}^3 = \frac{ab}{P_{\rm cr}} \tag{9}$$

$$V_{\rm cr} = 3b, \quad P_{\rm cr} = \frac{a}{27b^2}, \quad T_{\rm cr} = \frac{8a}{27Rb}$$

# Regular lattices and defects

Periodic solids

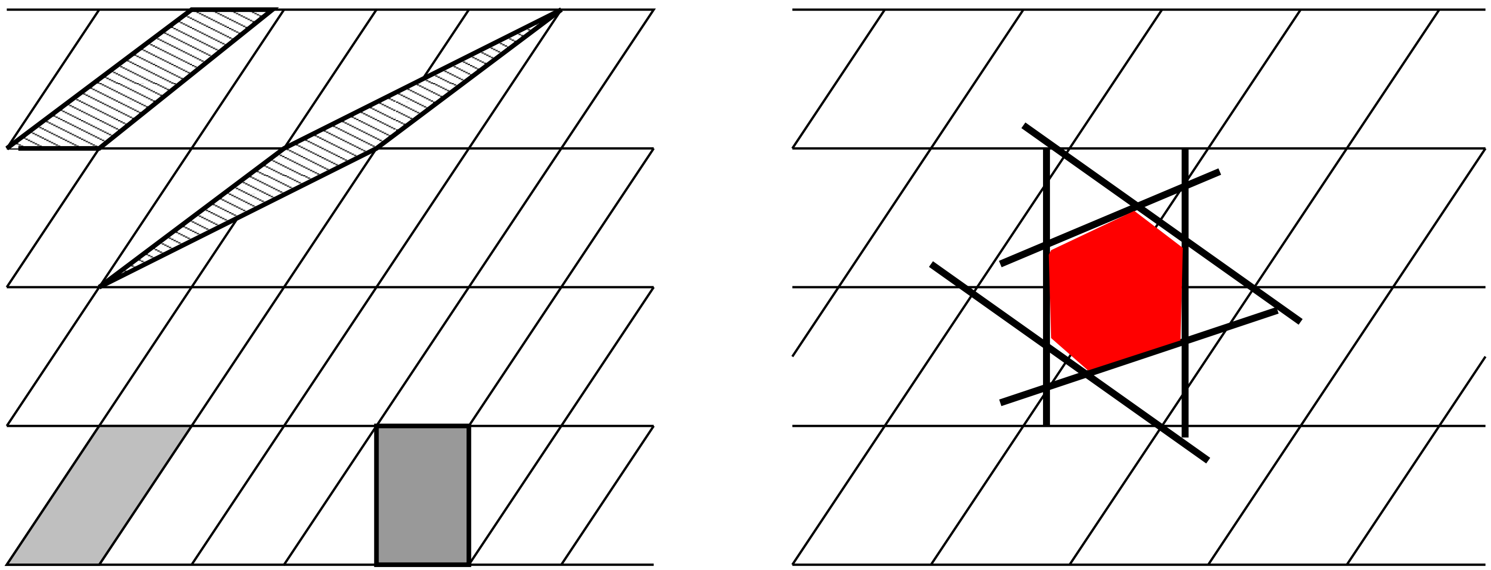
Non-periodic crystals

Sphere packing

Honeycomb

Bernard convection.

Defects of lattices: vacation, dislocation, disclination, monodromy defect.

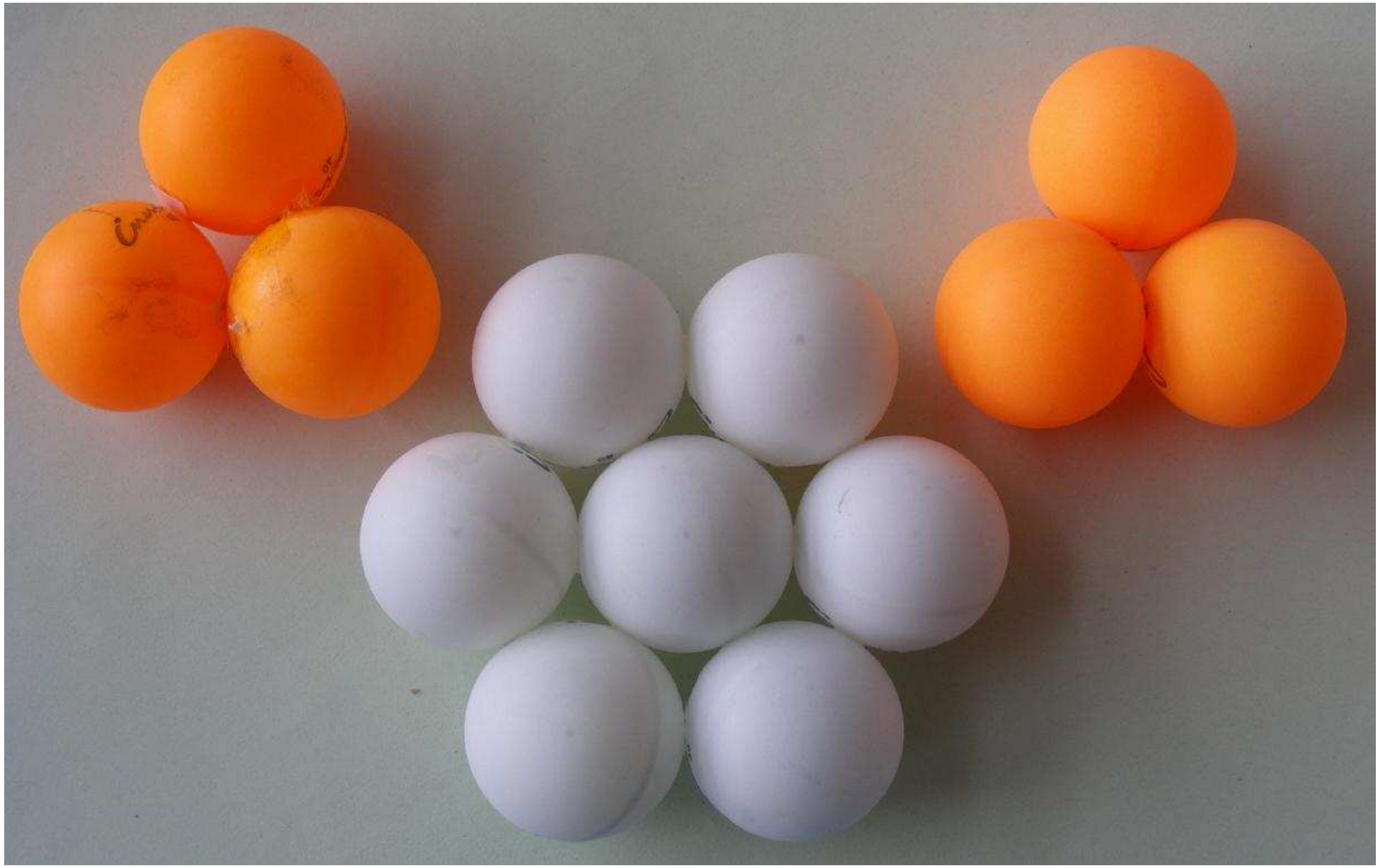


Different choices of an elementary cell of a lattice.

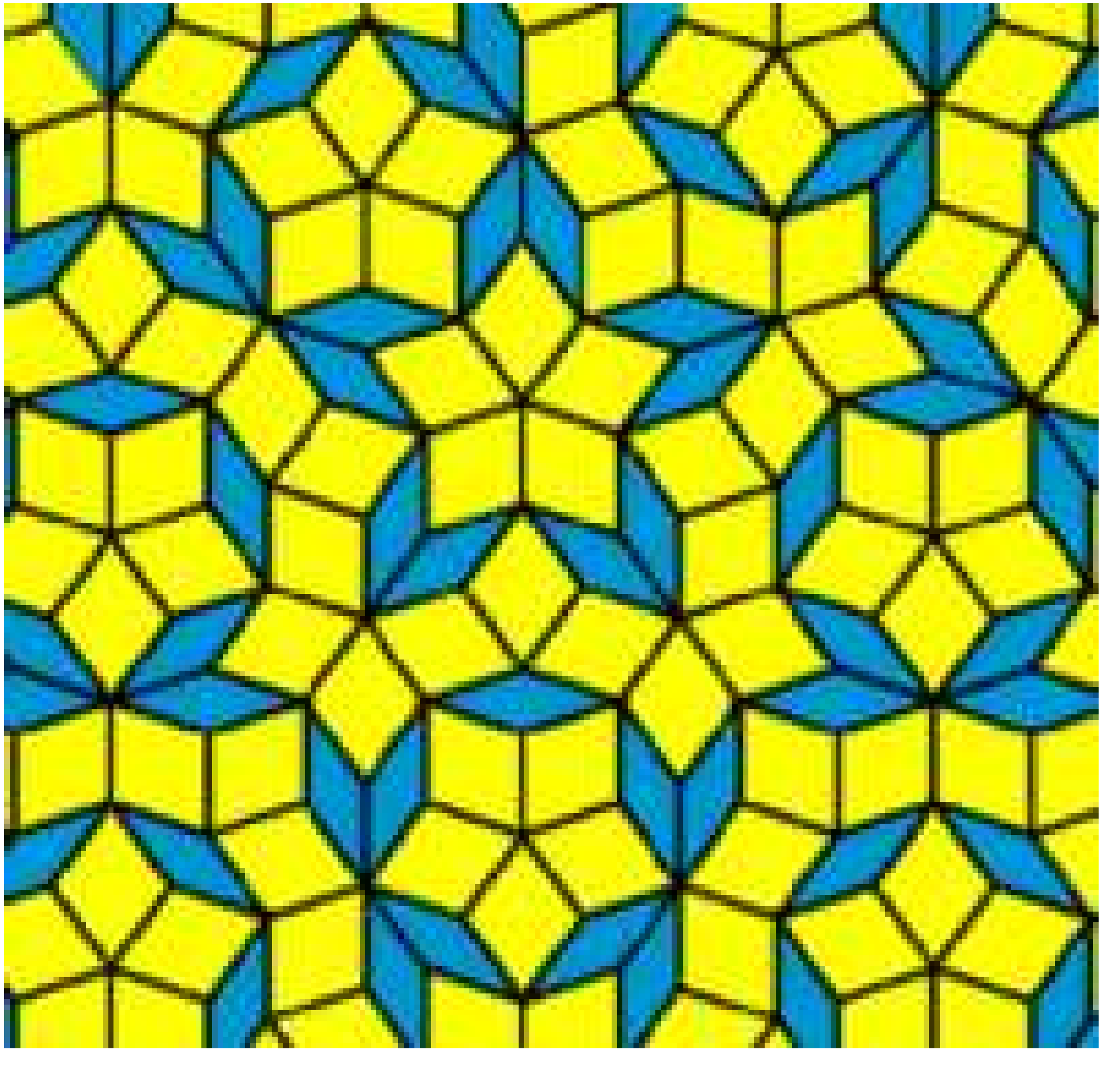

shutterstock


SILICONE OIL

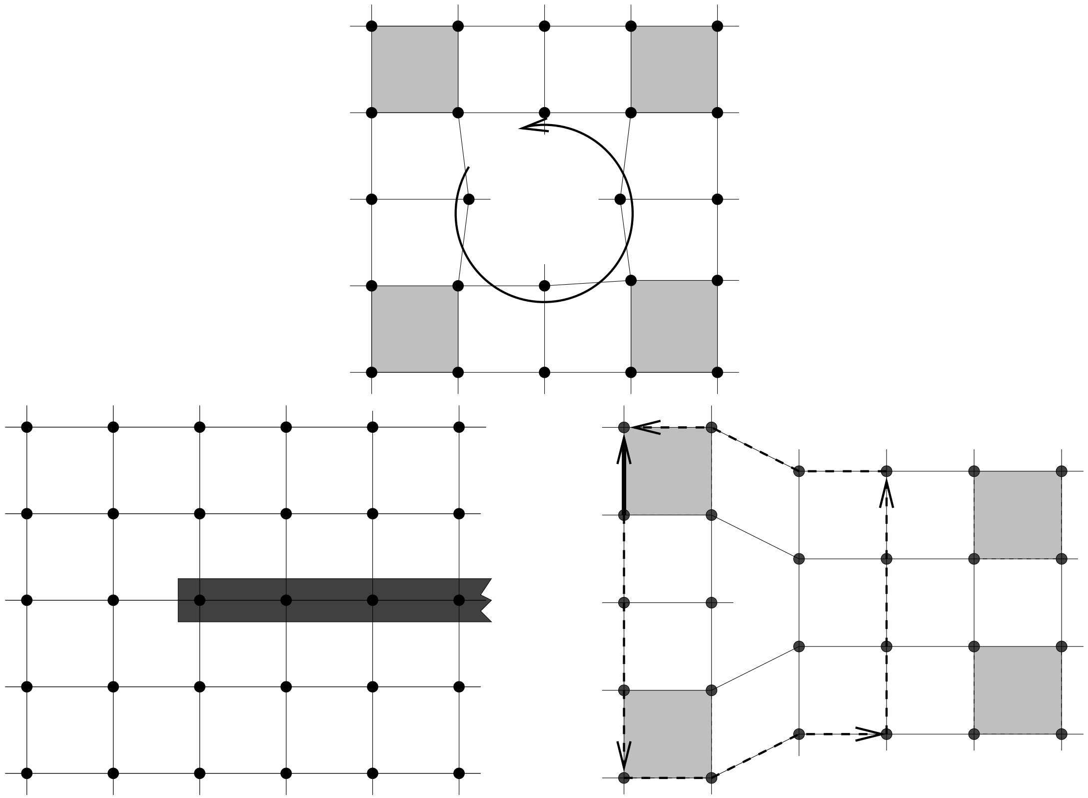
(G. Gaeta, Phys. Rep. 189, 4 (1990)) “Go to the kitchen, light up the stove, and put a pot on it (let us say about 30 cm diameter) with one cup of water and 3/4 cup of rice. The quantity of heat produced per unit time by the flame is your control parameter; in the beginning you keep it low, mix everything and wait for the system to reach equilibrium. Later you try to increase it. You observe that when the flame is high the rice gets into a peculiar pattern, with many holes; if you observe this pattern, you will notice a hexagonal symmetry. ... This example is physically relevant and has its own name, *Bérnard convection* (the convection cells are called Bérnard cells.) It takes place, besides in your kitchen, in stars. If you look at an astronomy book, you will probably find a picture of the Sun’s surface displaying the same kind of pattern that you observed in your overheated rice pot.“



The closest 2D-packing gives many alternative periodic and non-periodic packings.

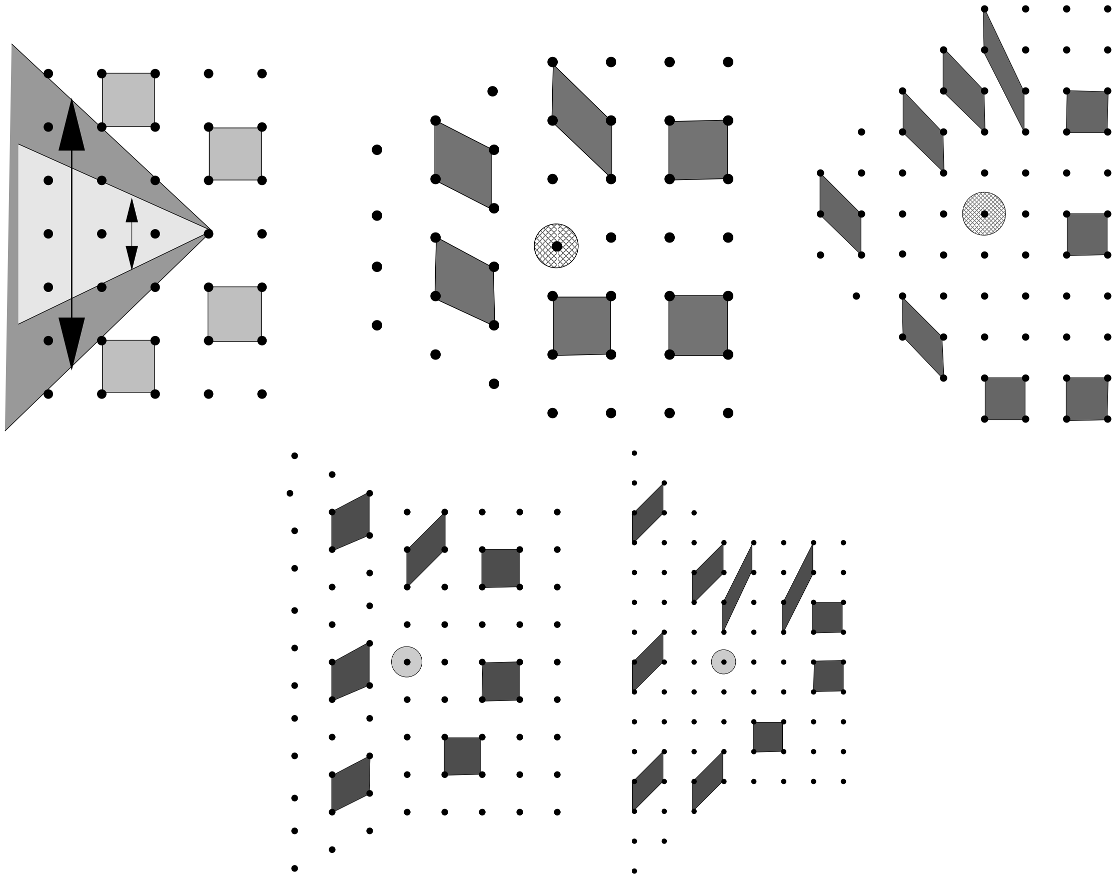


Penrose aperiodic tiling.

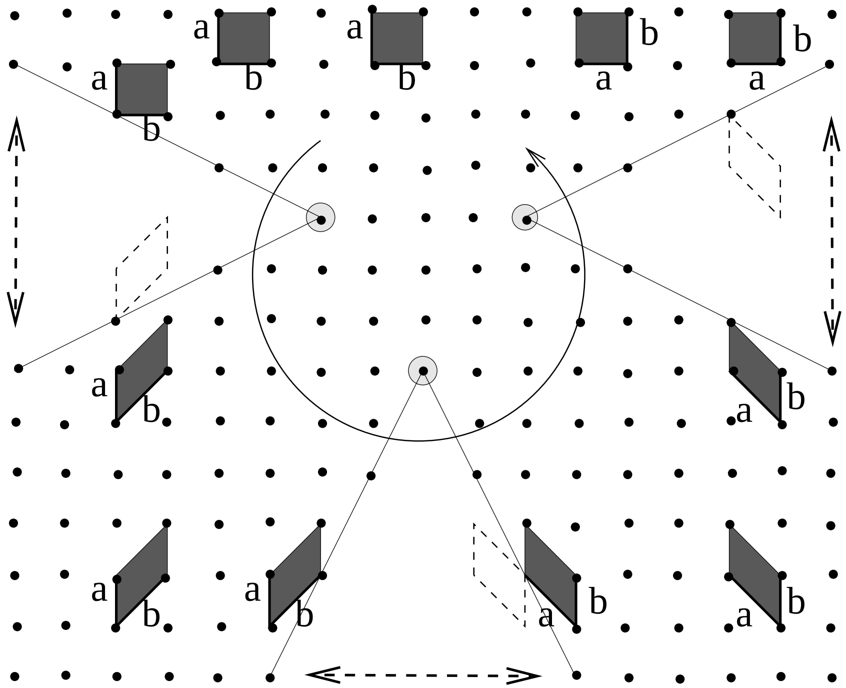


Lattice with vacation.

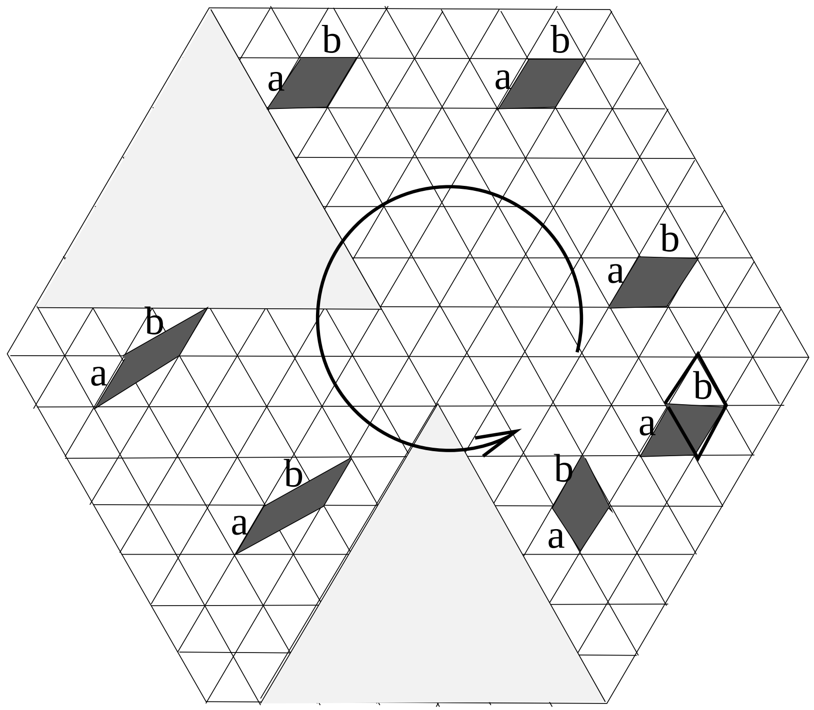
Construction of linear dislocation and lattice with linear dislocation



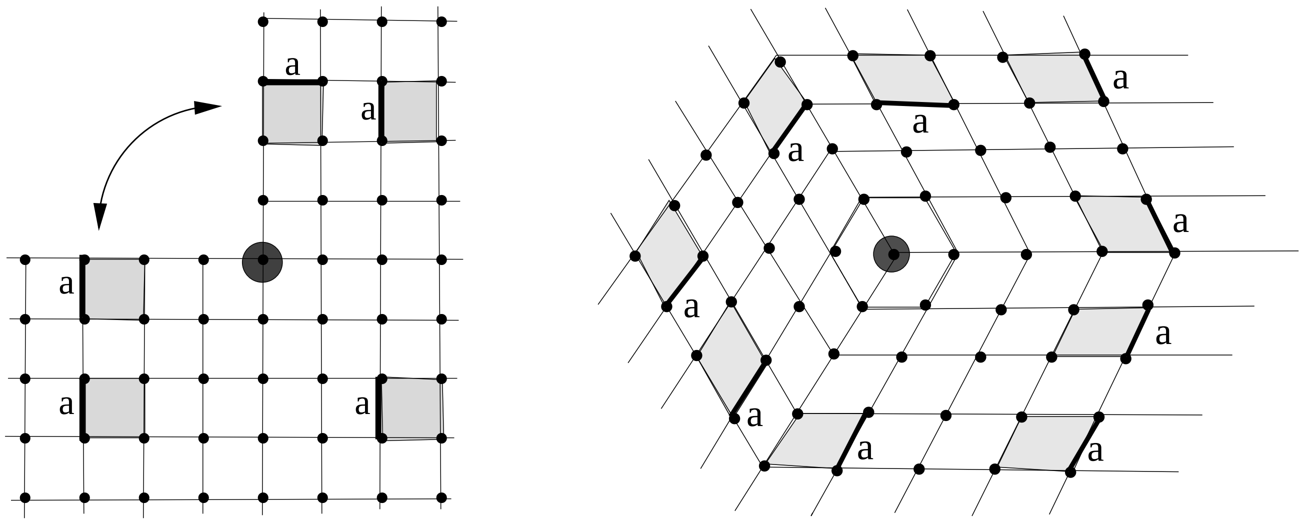
Construction of the angular dislocation (elementary monodromy defect).



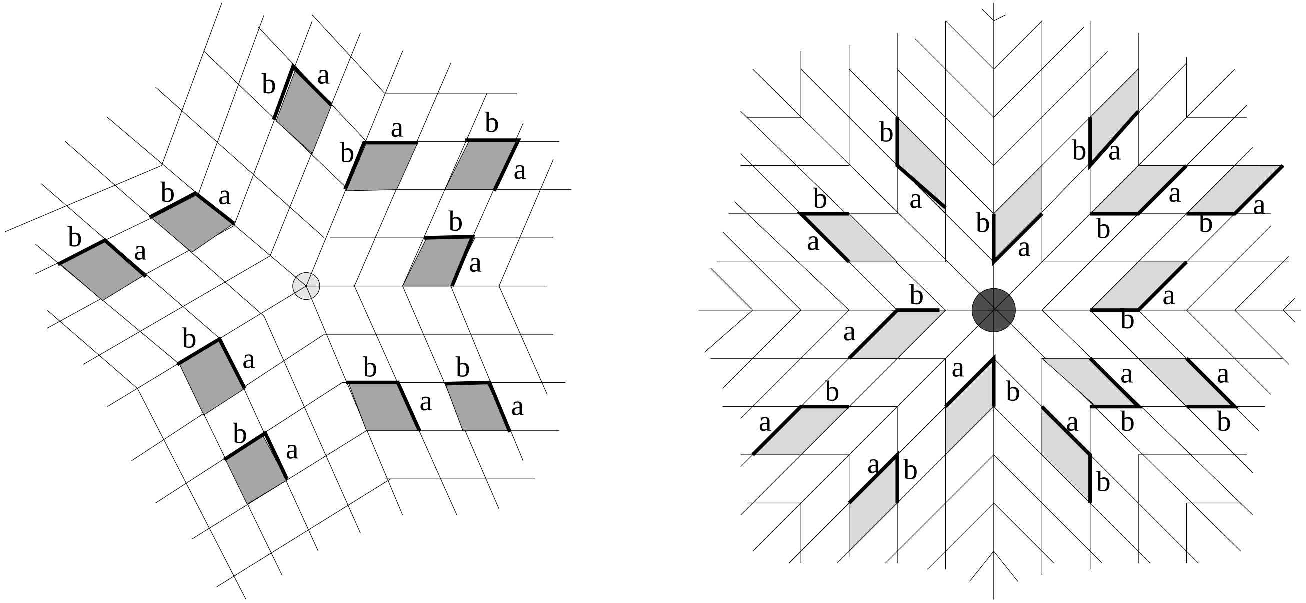

Regular square lattice with three elementary monodromy defects



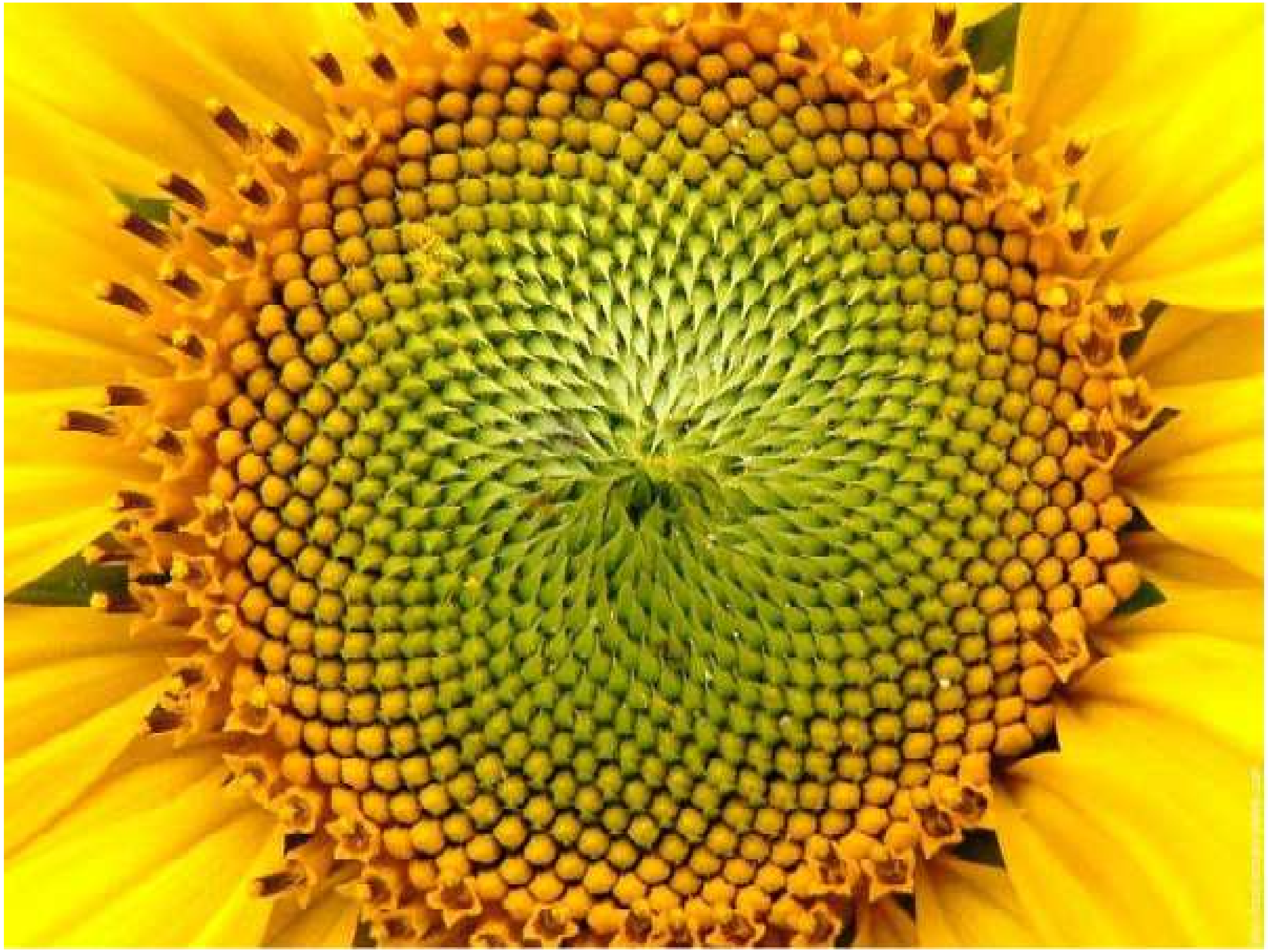

Regular triangular lattice with two elementary monodromy defects.

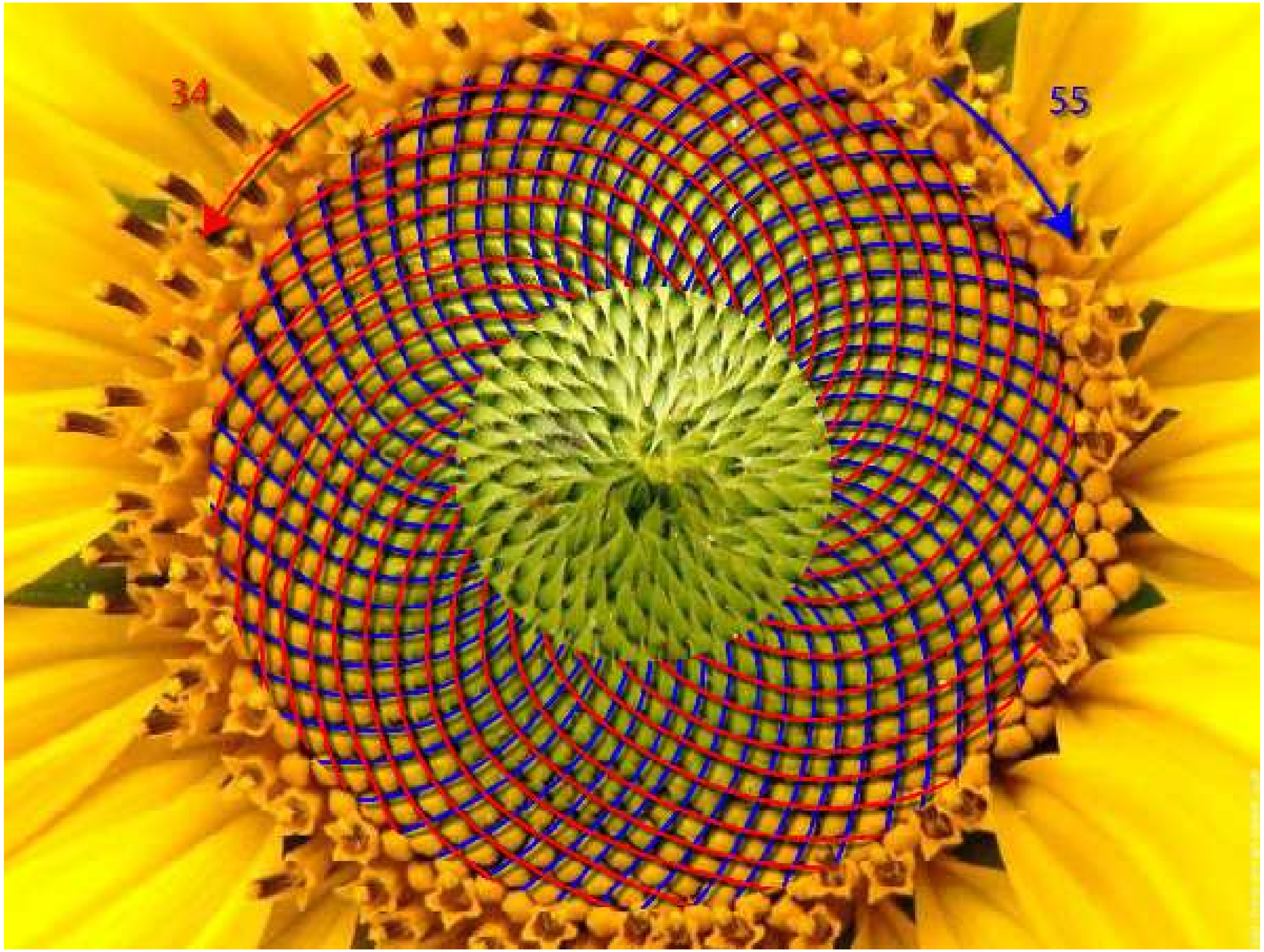



Construction of the rotational disclination by removing solid angle $\pi/2$ shown on the left picture.

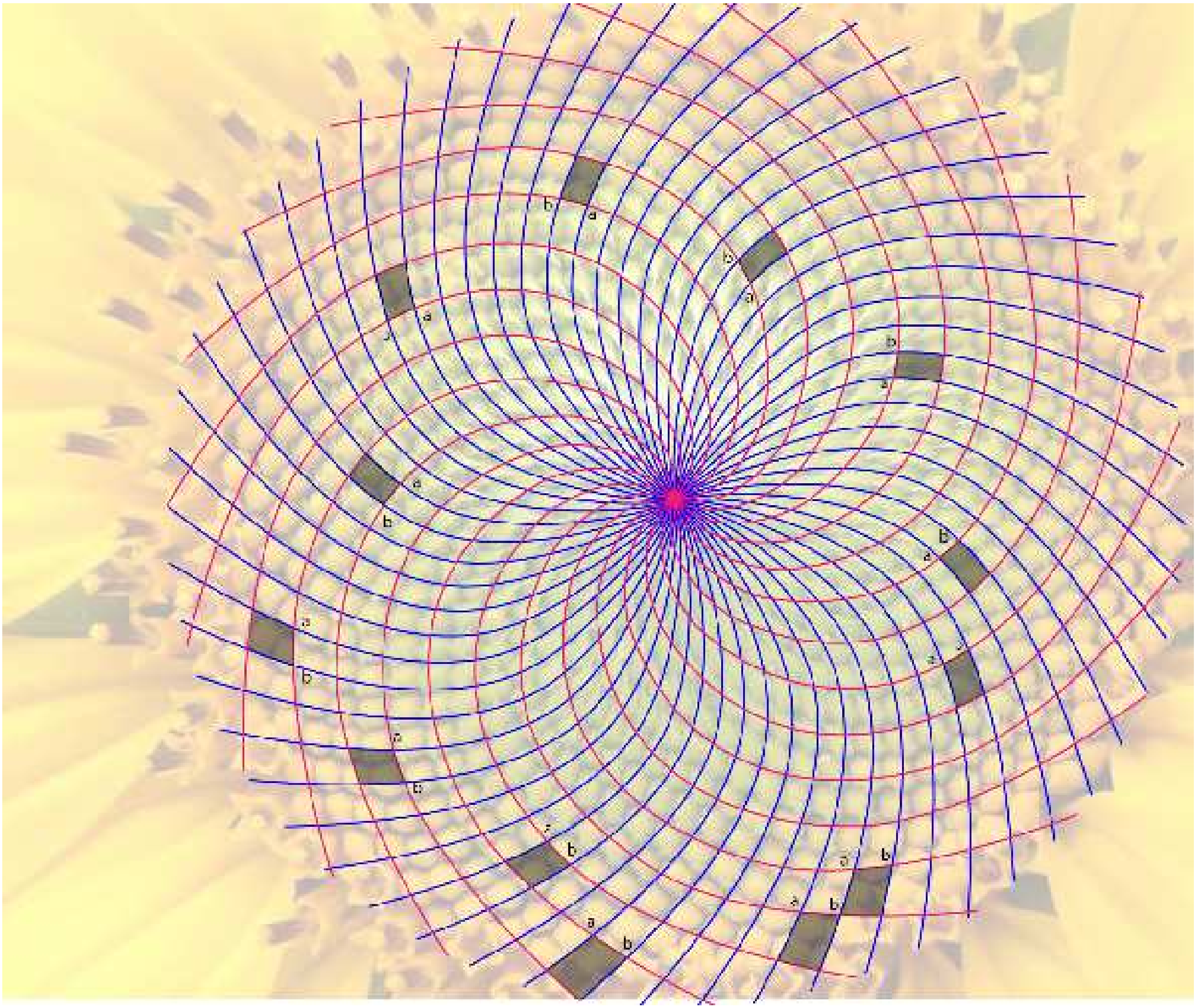


Construction of the rotational dislocation (disclination) by introducing solid angle $k\pi/2$. $k = 1$ on the left and $k = 4$ on the right picture.





34
55


a
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