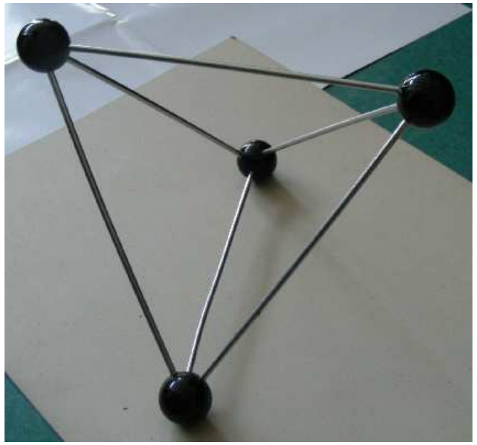
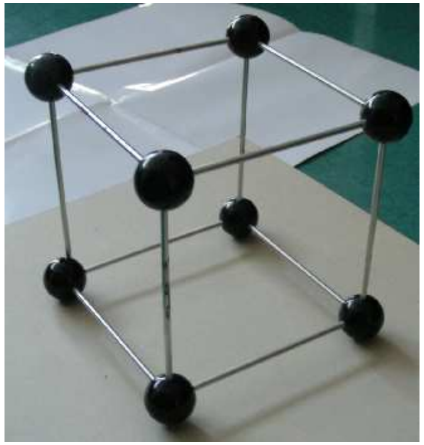
SYMMETRY

1. Symmetry operations.
2. Symmetry groups.
3. Group actions, invariants, covariants.
4. Generating functions applications.
5. Symmetry breaking and spontaneous symmetry breaking.
6. Curie principle.

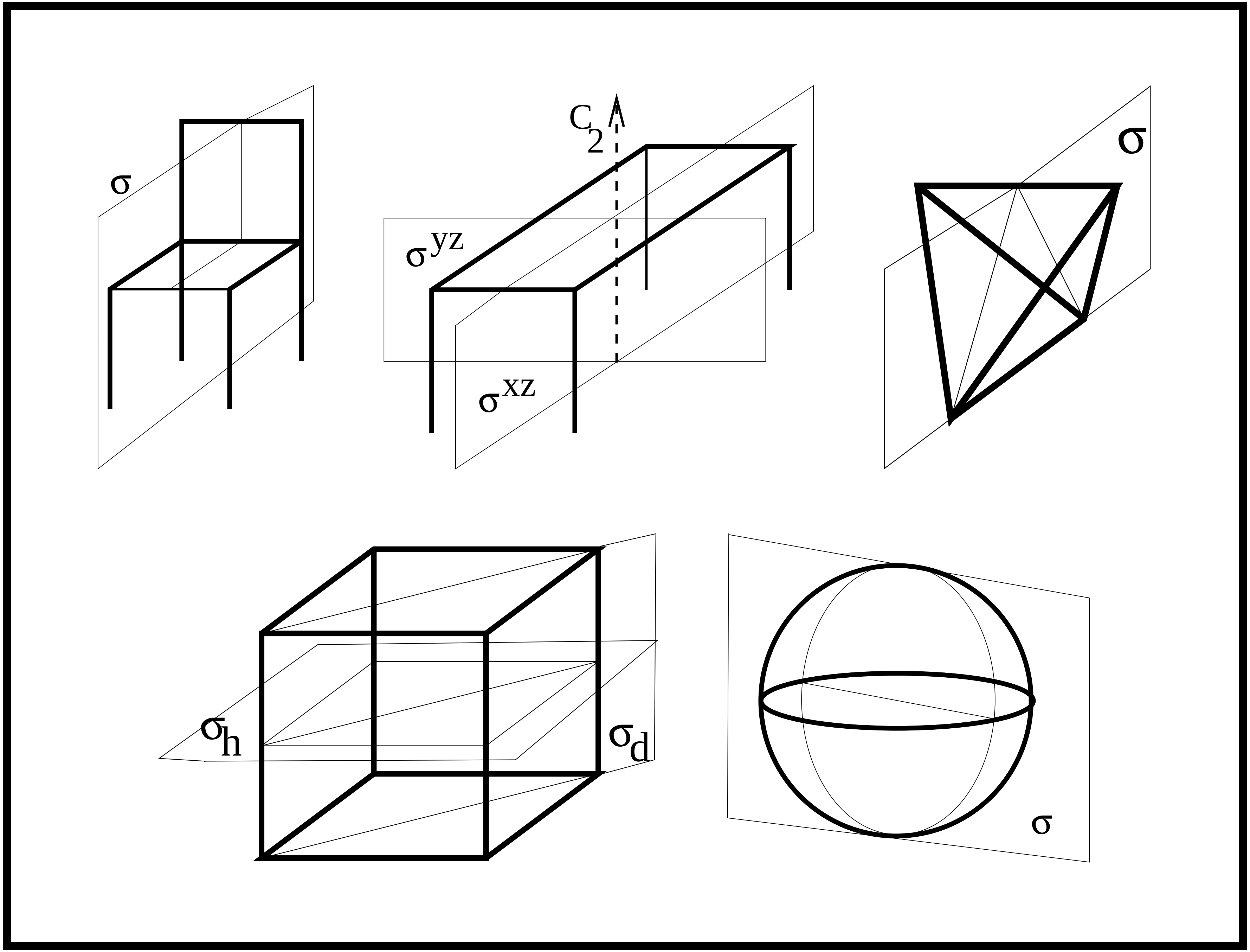
Symmetry operations

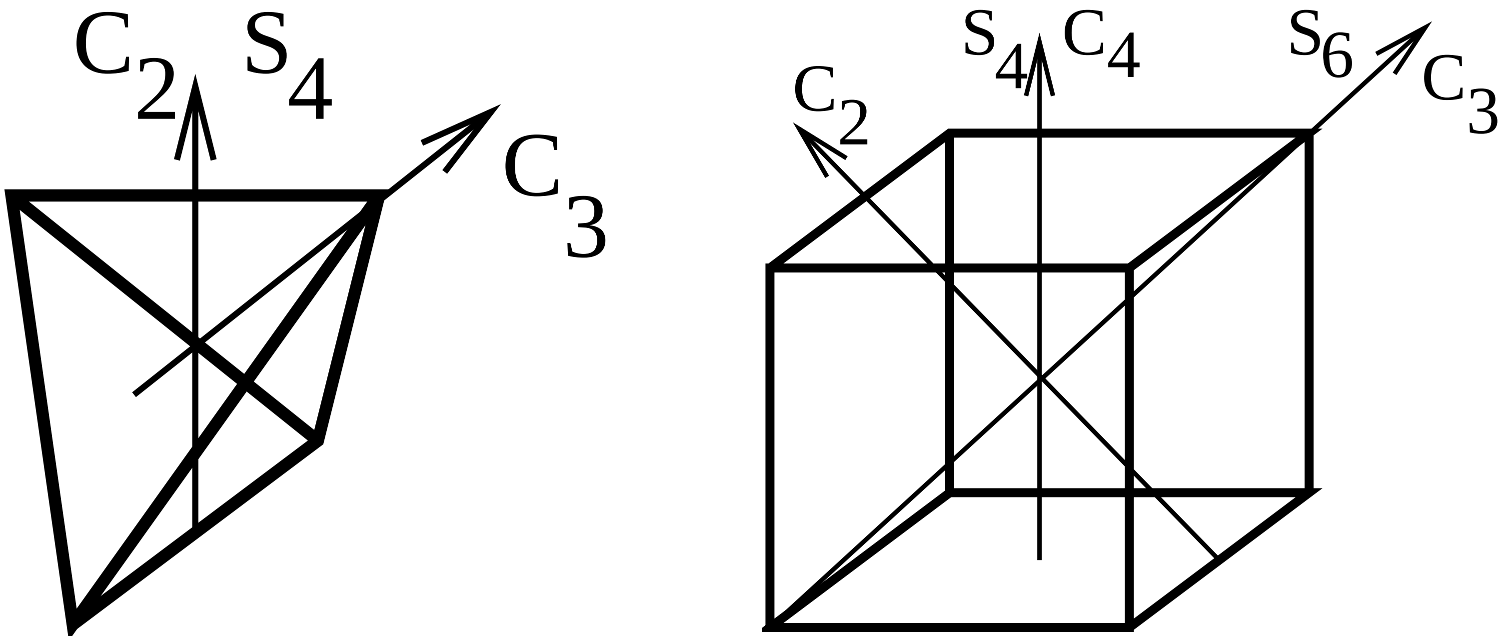
- Reflection, σ σ_h
- Rotation C_n - over $2\pi/n$
- Rotation-reflection S_n, $S_2 = i$, i - inversion.
- Translation $x \to x + \delta$
- Time reversal $t \to -t$
- Permutation of identical particles
- Color transformation
- Charge conjugation




Why some objects are more symmetric than others? Compare the chair, the table, the tetrahedron, the cube and the sphere.


σ
C2
σyz
σxz
σ
σh
σd
σ



Rotation and rotation–reflexion axes for tetrahedron et for cube.

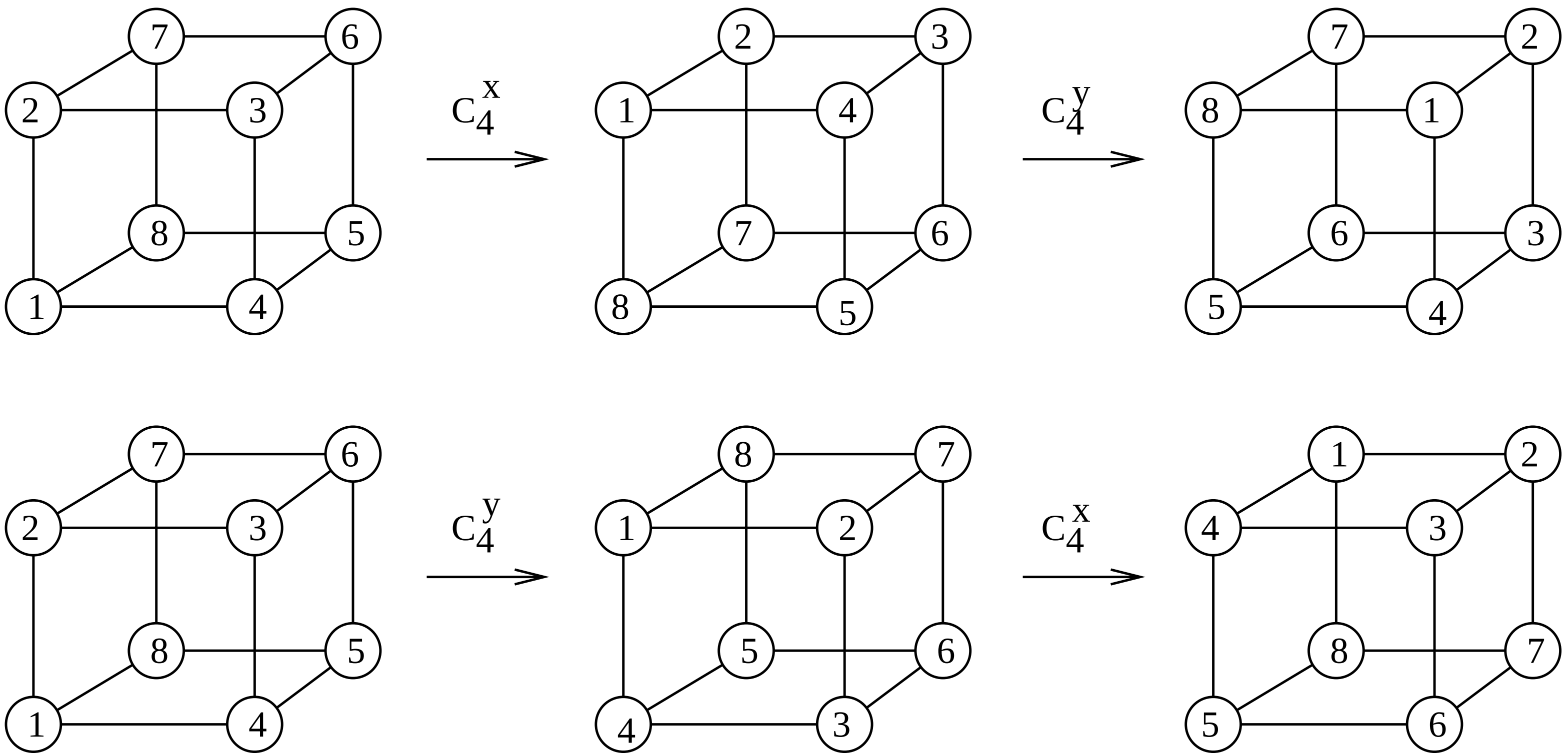
Composition of symmetry operations

$$C_n \cdot C_n = C_n^2$$

$$(C_2)^2 = E, \quad (C_3)^3 = E, \quad (C_n)^n = E$$

$$\sigma^2 = E, \quad (S_2)^2 = i^2 = E$$

$$C_n^{n-1} = C_n^{-1}$$



The operations C_4^x and C_4^y does not commute.

GROUP

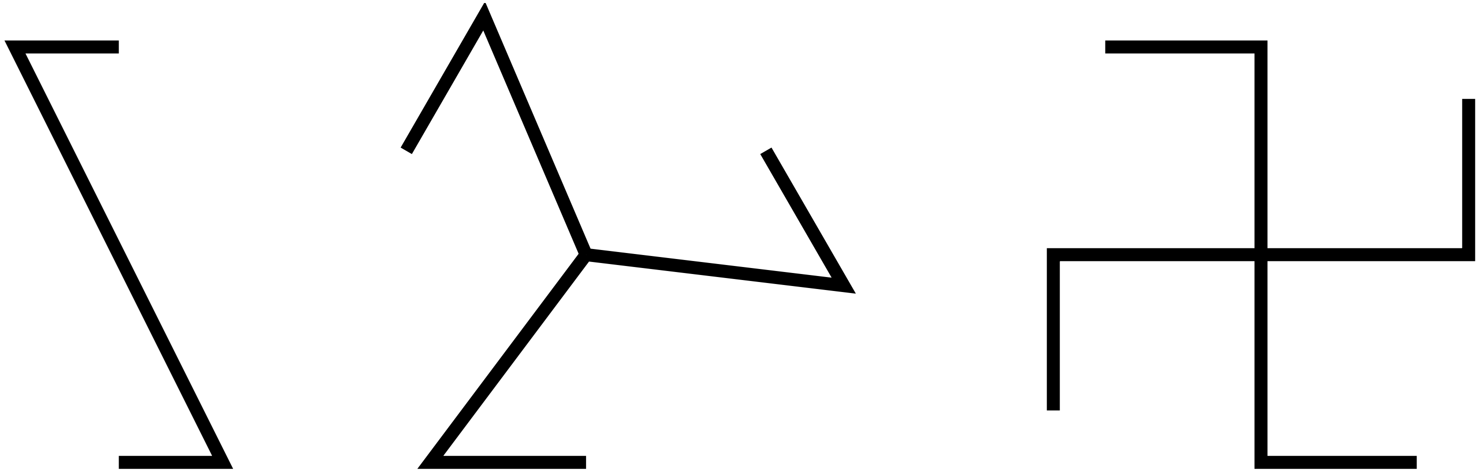
A group is a set of elements named symmetry operations

$$G : \{ \, g_1 = E, g_2, \ldots, g_N \},$$

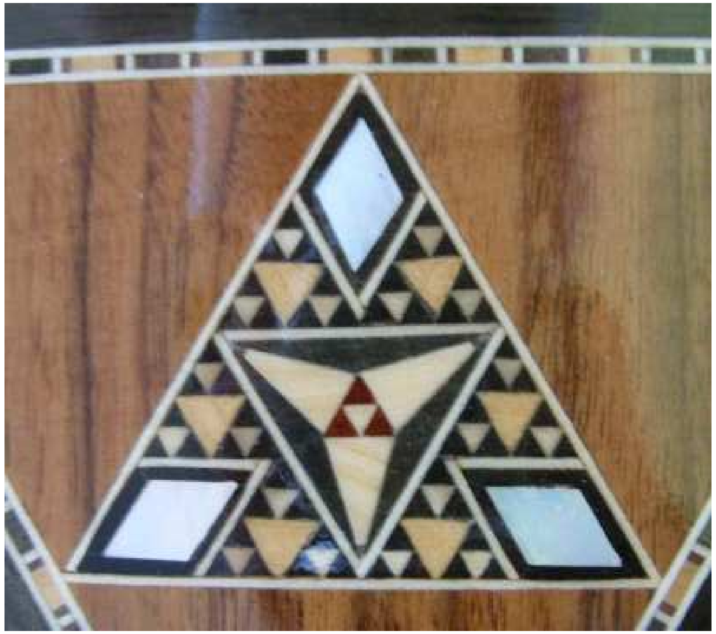
with the law of composition :

1. $g_i g_j$ is defined and belongs to G,
2. the law of composition is associative, $(g_i g_j) g_k = g_i (g_j g_k)$, but in general it is not commutative $g_i g_j \neq g_j g_i$,
3. there exists an element unity, E, such that $E g_i = g_i = g_i E$, for all $g_i \in G$,
4. each element $g_i \in G$ has an inverse element $(g_i)^{-1}$, $g_i (g_i)^{-1} = E = (g_i)^{-1} g_i$.

Figure with $\mathbf{C}_s$ symmetry used in psychological tests.



Figures with 2D-symmetry group $\mathbf{C}_n$, $n = 2, 3, 4$



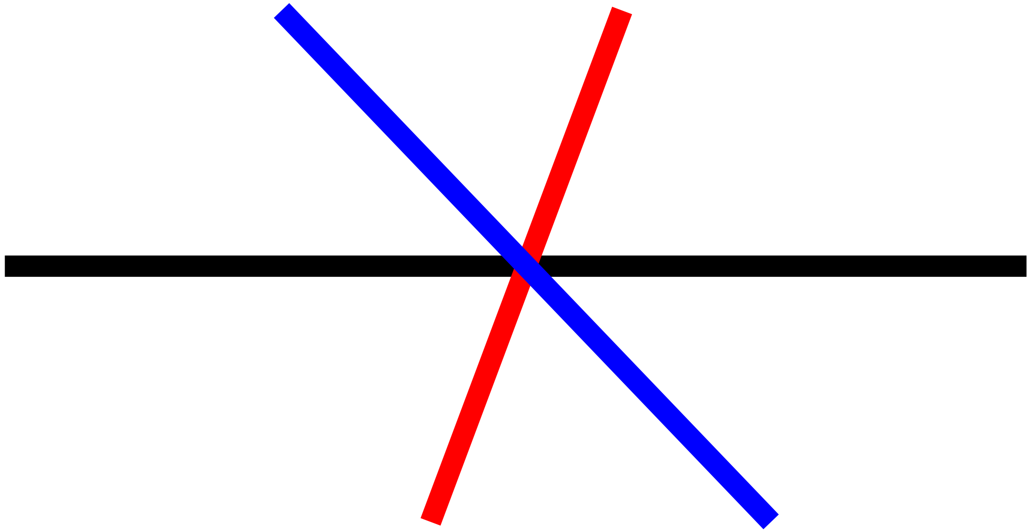
D_3

D_4

Examples of 2D-symmetry groups.

D_1

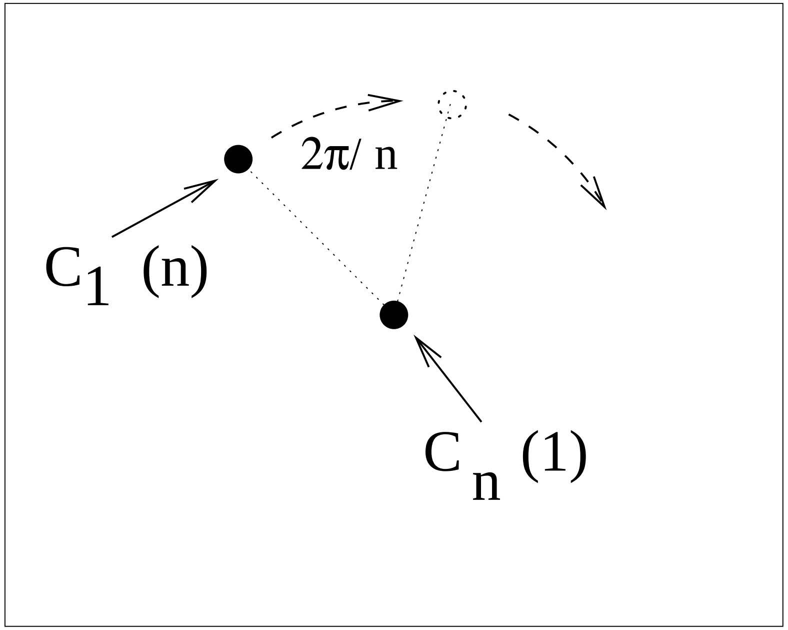
D_{13}



C_i

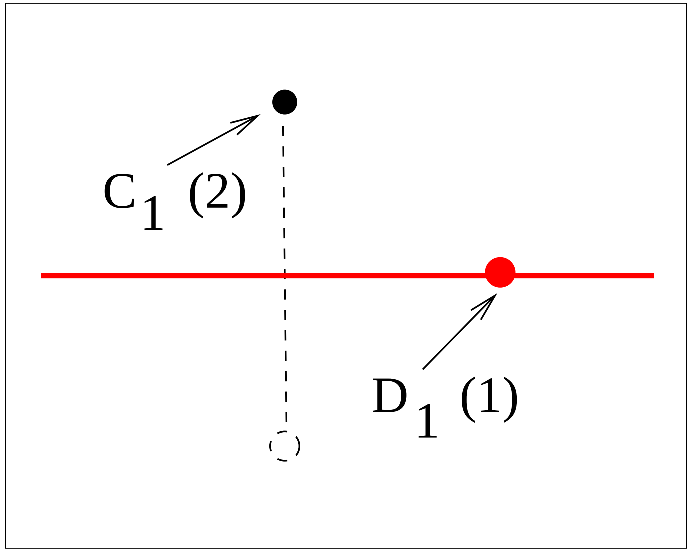
3-D symmetry group.

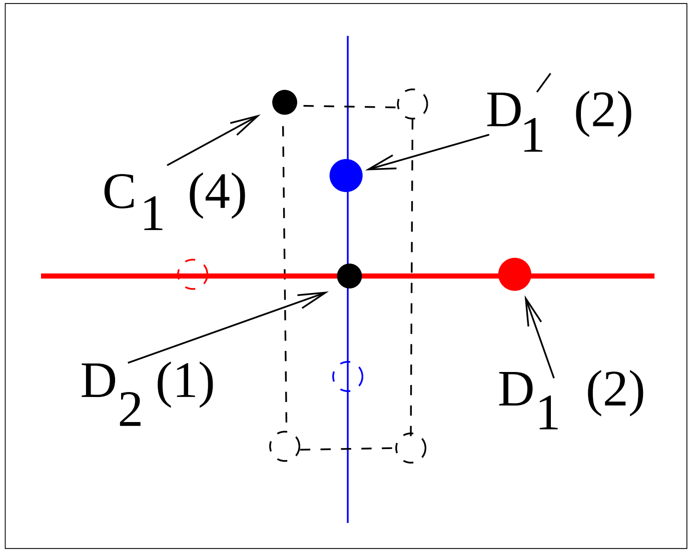
Group action. Orbits. Stabilizers.



Action of group C_n on 2-D space.

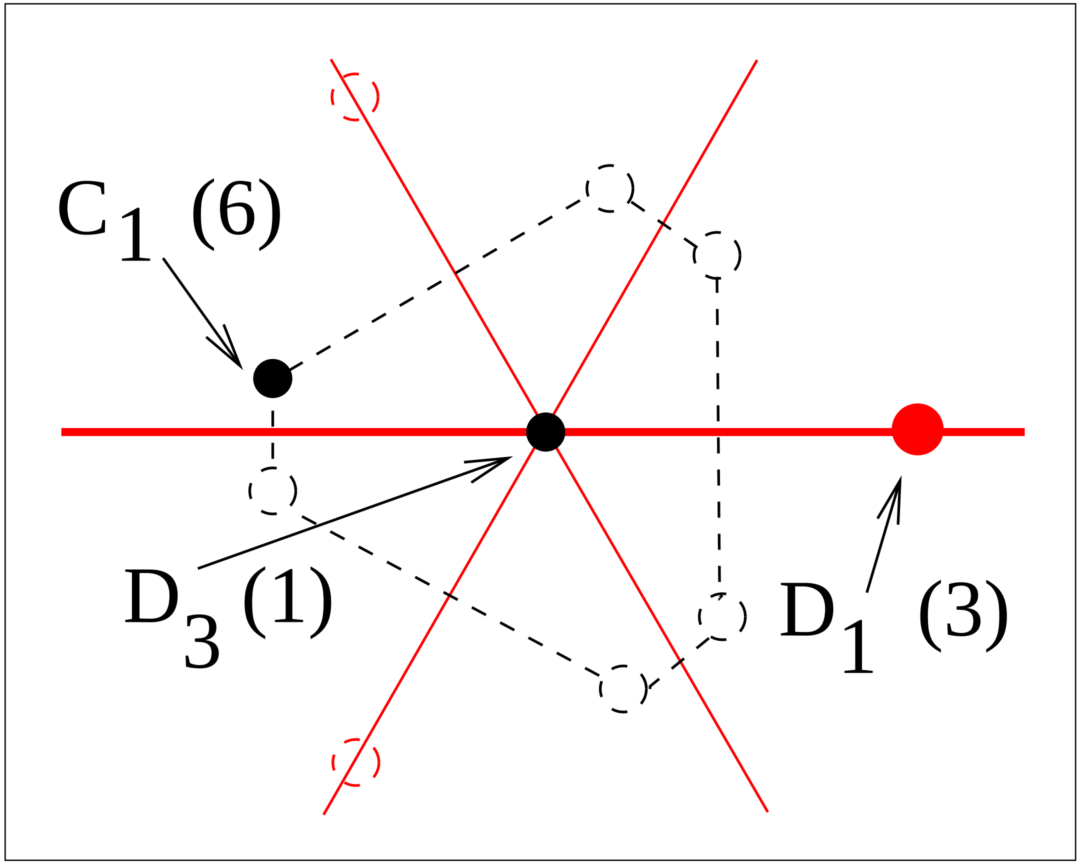
The stabilizer and the number of points in orbit are indicated.



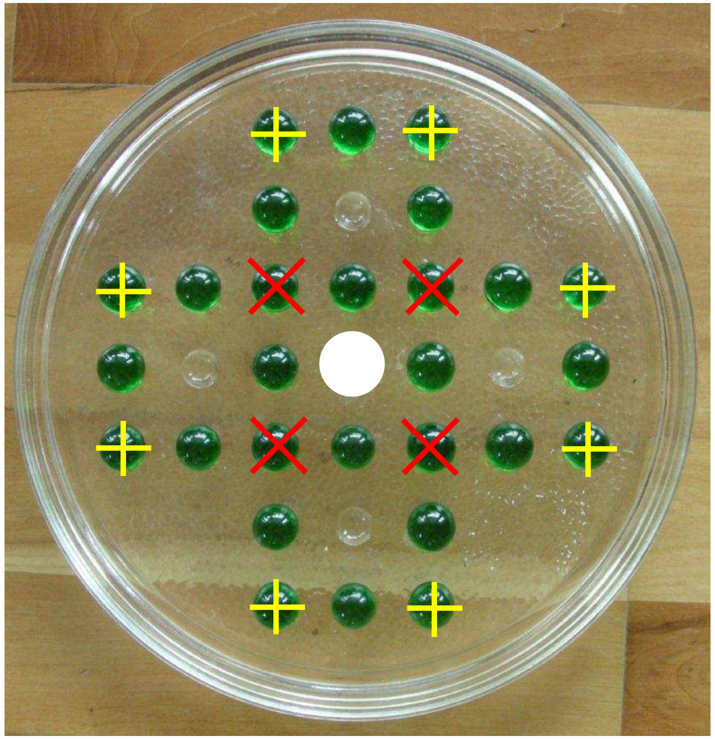


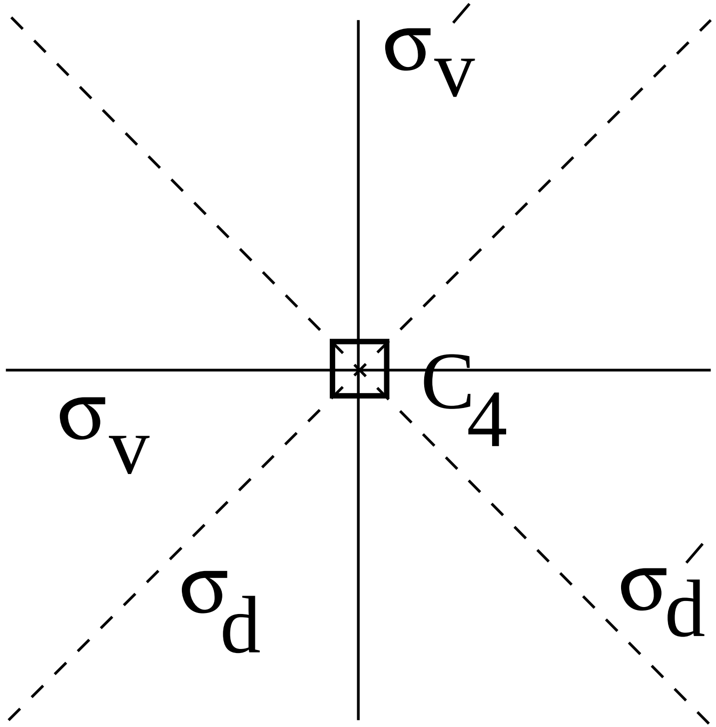
D_1 action

D_2 action



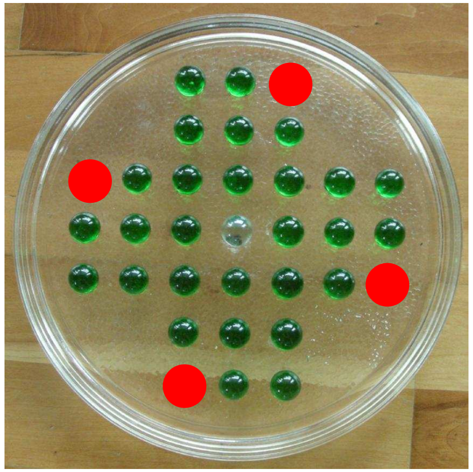
D_3 action



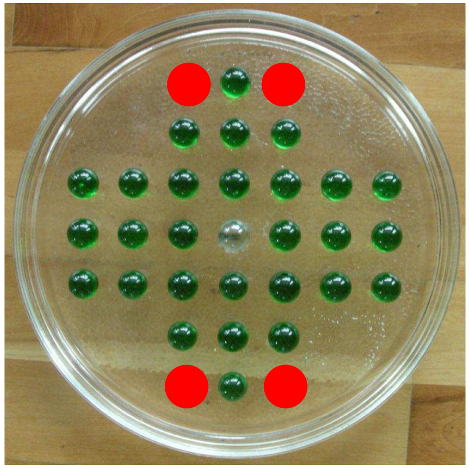


Orbits of $\mathbf{C}_{4v}$ (D_4) action.

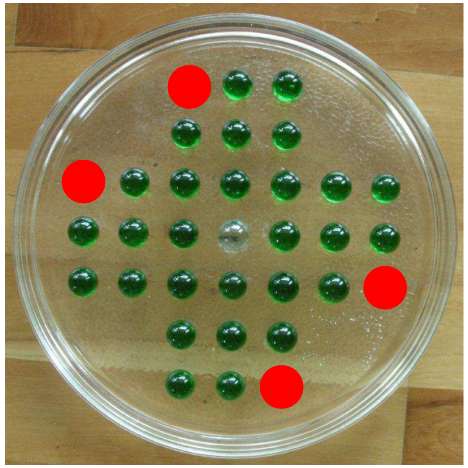
Stabilizer	C_{4v}	-	1 point
Stabilizer	C_s	-	4 points
Stabilizer	C'_s	-	4 points
Stabilizer	C_1	-	8 points



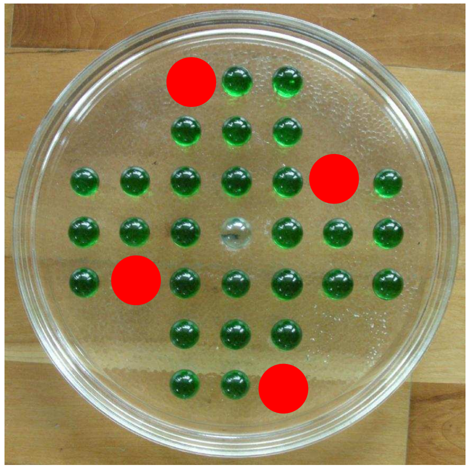
$\mathbf{C}_4$



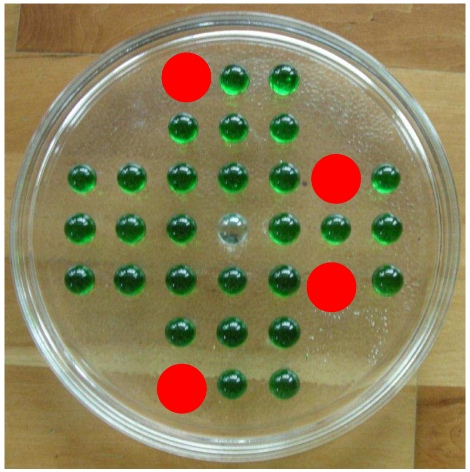
$\mathbf{C}_{2v}$



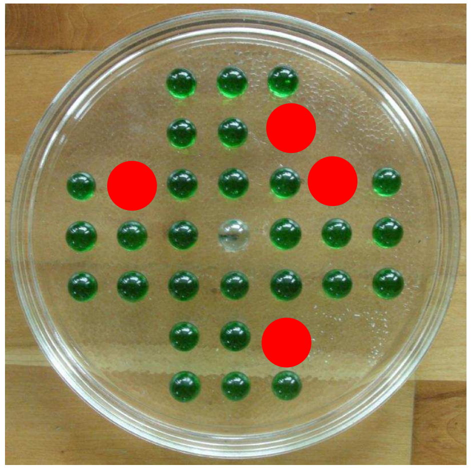
$\mathbf{C}'_{2v}$



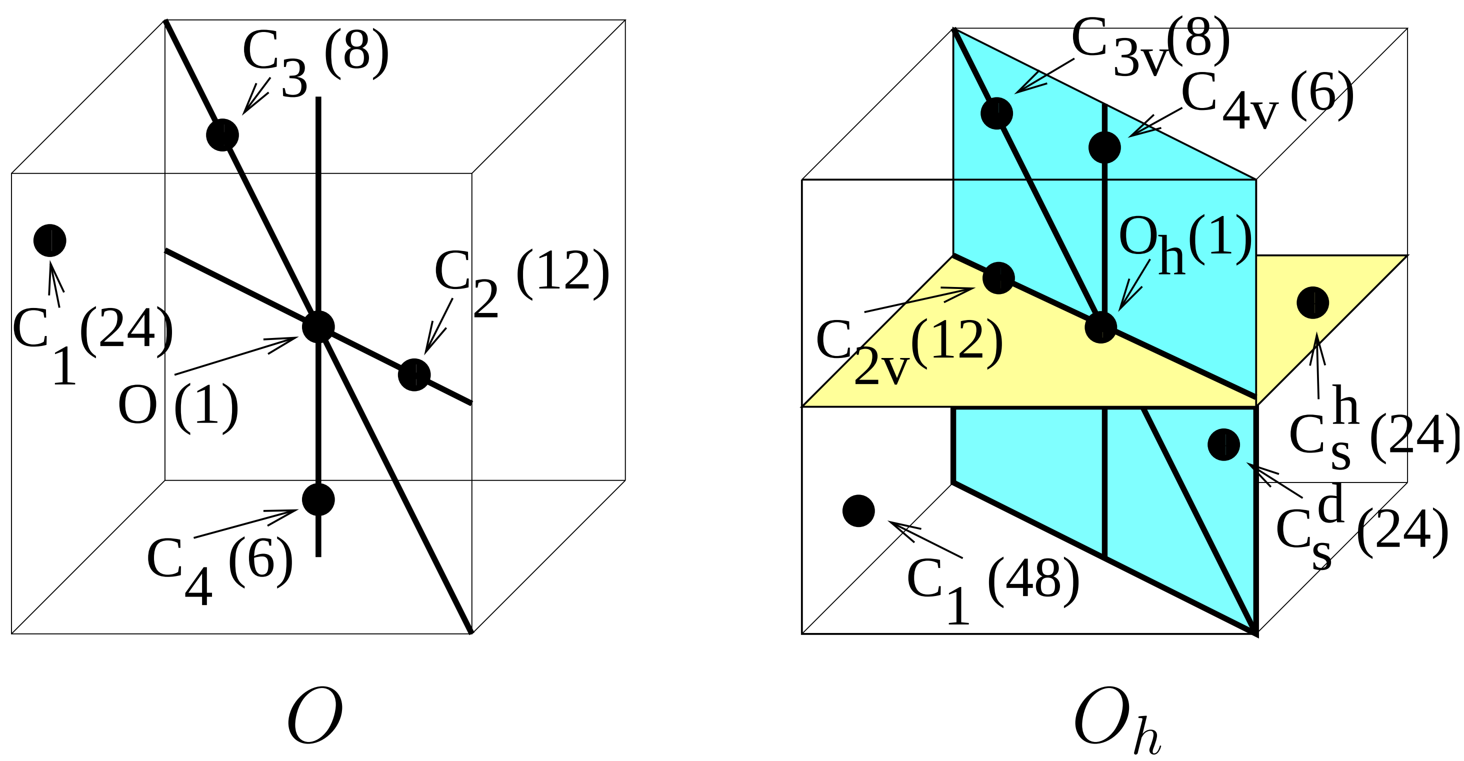
$\mathbf{C}_2$



$\mathbf{C}_s$



$\mathbf{C}'_s$



C3 (8)
C2 (12)
C1 (24)
O (1)
C4 (6)
O
C3v (8)
C4v (6)
Oh (1)
C2v (12)
Cs h (24)
Cs d (24)
C1 (48)
Oh

List of 3-D finite point groups

$$\mathbf{C}_n, \quad \mathbf{S}_{2n}, \quad \mathbf{C}_{nv}, \quad \mathbf{C}_{nh}, \quad \mathbf{D}_n, \quad \mathbf{D}_{nd}, \quad \mathbf{D}_{nh},$$

$$\mathbf{T}, \quad \mathbf{T}_d, \quad \mathbf{T}_h, \quad \mathbf{O}, \quad \mathbf{O}_h, \quad \mathbf{I}, \quad \mathbf{I}_h$$

Dynamical applications of orbits and strata

Stratum - collection of orbits of the same type (with equivalent stabilizer).

Theorem (Michel, 1971). *In the smooth action of a compact (or finite) group G on a finite-dimensional manifold M, the gradient of every G-invariant functions vanishes on the orbits which are isolated in their strata. These orbits are called* critical.

Orbits and strata for the action of O_h symmetry group on the two-dimensional sphere.

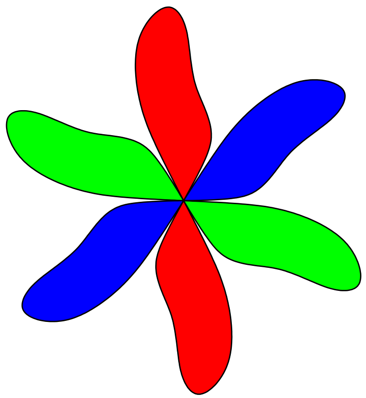
Stabilizer	Number of points per orbit	Number of orbits per stratum	Comments
C_{4v}	6	1	Critical
C_{3v}	8	1	Critical
C_{2v}	12	1	Critical
C_s	24	∞	Open
C_s'	24	∞	Open
C_1	48	∞^2	Generic

Exercises

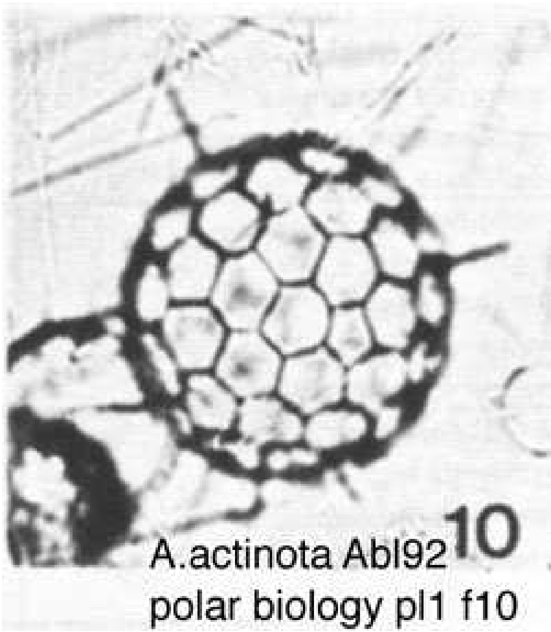
1. What molecules among AB_3, A_2B_3, A_3B_3, A_4B_3, A_5B_3, A_6B_3 can be characterized by the equilibrium configuration of symmetry $\mathbf{C}_i$?

2. Can equilibrium configuration of AB_3 molecule be characterized by symmetry group $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$, $\mathbf{C}_4$, $\mathbf{C}_s$, $\mathbf{C}_i$, $\mathbf{C}_{2v}$, $\mathbf{C}_{2h}$, $\mathbf{C}_{3v}$, $\mathbf{C}_{4v}$, $\mathbf{D}_3$, $\mathbf{D}_{3h}$, $\mathbf{T}_d$?

3. Find the symmetry operations for this artificial flower



4. Radilaria are found as zooplankton throughout the ocean. There are a number of species whose skeleton is formed by hexagons and pentagons. Find the number of pentagons.



1. Is it possible to cover the surface of the sphere with hexagons only?
2. Let the lattice on the sphere be formed by hexagons, pentagons, and heptagons. Find restrictions on the possible number of different polygons.

5. Let us suppose that instead of sphere we have a torus.

   1. Is it possible to construct on the surface of torus a lattice formed by hexagons only?
   2. If the lattice on the surface of torus is formed by pentagons, hexagons, and heptagons what is the restriction on the number of different polygons?

6. Find the number of hexagons and pentagons for the Morocco‘s wooden toy.

Generating functions for invariants

1-D symmetry group $\quad x \longrightarrow -x$

Variable x is not invariant under the group action.

x^2 is invariant under the group action.

What is the number of invariants in each degree, that can be constructed?

The obvious answer can be written in terms of generating function

$$\frac{1}{1-t^2} = \sum_{n=0}^{\infty} t^{2n} = 1 + t^2 + t^4 + \dots$$

There is one invariant in each non-negative even degree.
Arbitrary invariant can be written as a polynomial of x^2: $\mathcal{P}(x^2)$

3-D inversion symmetry group $\{x, y, z\} \longrightarrow \{-x, -y, -z\}$.

It is possible to construct 6 quadratic invariants :

$$x^2, y^2, z^2, xy, yz, xz.$$

Generating function for invariants:

$$\frac{1+3t^2}{(1-t^2)^3} = 1 + 6t^2 + 15t^4 + 28t^6 + \dots$$

or in more detailed form depending on three auxiliary variables

$$\frac{1 + t_1 t_2 + t_1 t_3 + t_2 t_3}{(1-t_1^2)(1-t_2^2)(1-t_3^2)}$$

Arbitrary invariant polynomial has the form :

$$\mathcal{P}_0(x^2, y^2, z^2) + xy\mathcal{P}_1(x^2, y^2, z^2) + yz\mathcal{P}(x^2, y^2, z^2) + xz\mathcal{P}(x^2, y^2, z^2)$$

O_h cubic group action on three spatial variables x, y, z.

$$\frac{1}{(1-t^2)(1-t^4)(1-t^6)}$$

Symbolic meaning : there are three invariant polynomials, one θ_2 of degree 2, one θ_4 of degree 4, and one θ_6 of degree 6.

Arbitrary invariant is a polynomial $\mathcal{P}(\theta_2, \theta_4, \theta_6)$

Examples of group actions and spaces of orbits.

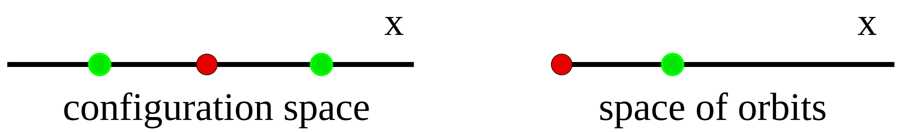
One-dimensional motion under presence of Z_2 symmetry.

Symmetry group action : $(x) \to (-x)$.

In coordinate space $x = 0$ is the only one-point orbit. It is *critical.*

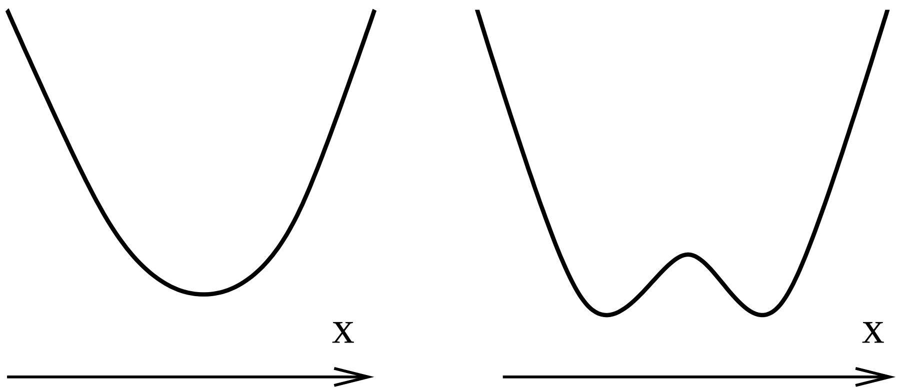
All other orbits includes two points $\pm x_0$ with $x_0 \neq 0$.

Space of orbits is a ray.



$x = 0$ is always a stationary point. If there is stationary point at $x \neq 0$ there are necessarily two equivalent stationary points.

Going from $x = 0$ minimum to minimum at $x \neq 0$ corresponds to spontaneous symmetry breaking phenomenon.



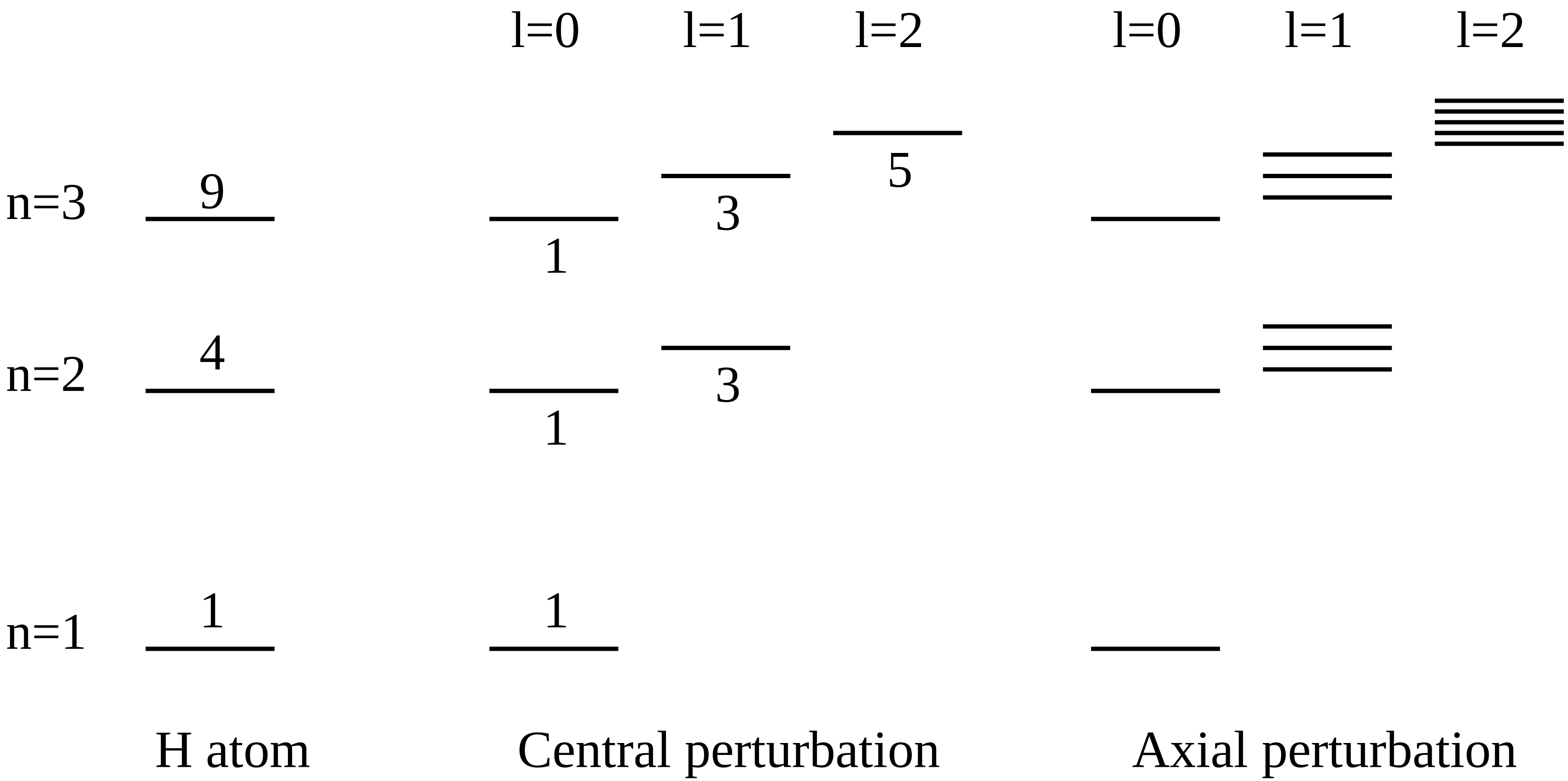
Characteristic features of **spontaneous symmetry breaking**.

Symmetry of the problem is not changed.

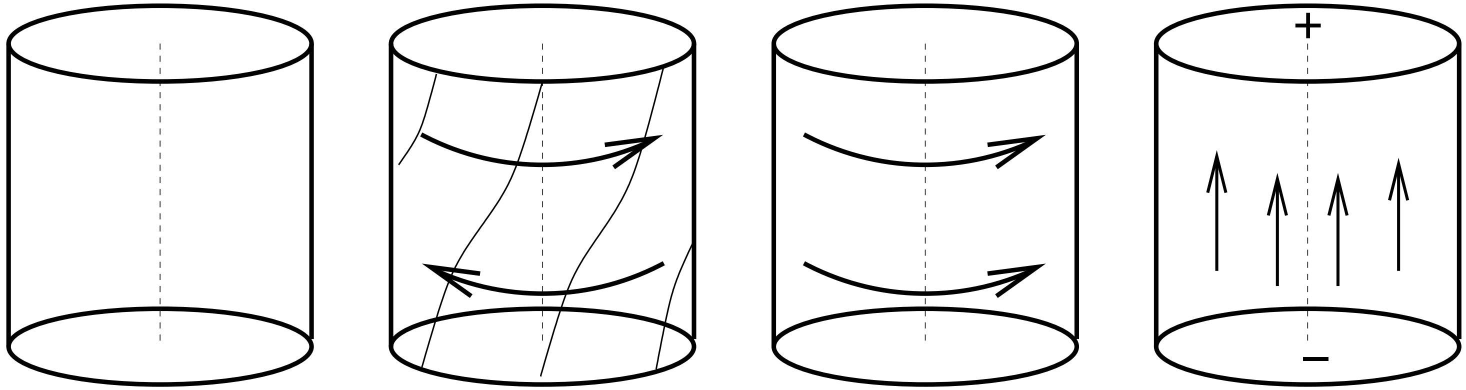
Symmetry of solution is changed (decreased).

The number of solutions increases.

Political cartoon ca. 1900, showing the United States Congress as Buridan’s ass, hesitating between a Panama route or a Nicaragua route for an Atlantic-Pacific canal.



Schematic representation of the degeneracy splitting of hydrogen atom levels due to symmetry breaking.



$\mathbf{D}_{\infty h}$ $\mathbf{D}_{\infty}$ $\mathbf{C}_{\infty h}$ $\mathbf{C}_{\infty v}$

Examples of realization of axial symmetry groups.

Non-deformed cylinder - $\mathbf{D}_{\infty h}$; twist cylinder - $\mathbf{D}_{\infty}$;

rotating cylinder - $\mathbf{C}_{\infty h}$; cylinder with axis asymmetry - $\mathbf{C}_{\infty v}$

Electric field - $C_{\infty v}$ *Magnetic field* - $C_{\infty h}$

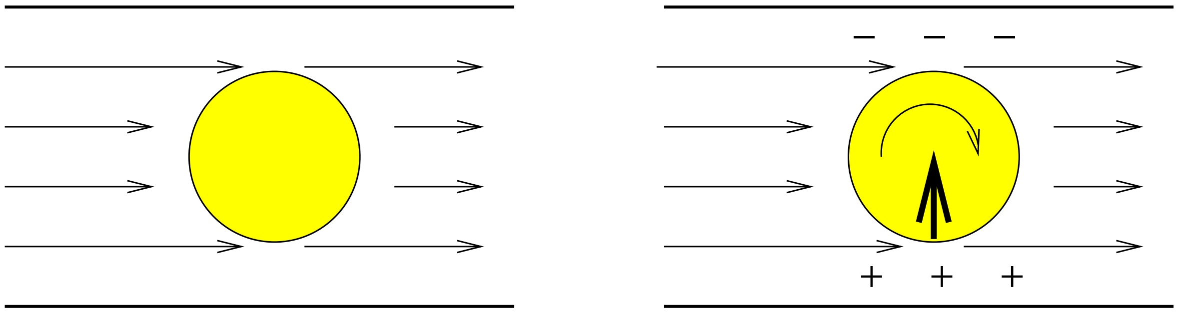


Table 1: Analogy between Magnus effect and Hall effect.

Hydrodynamics	Electromagnetism
Rotating cylinder	Magnetic field
Uniform hydrodynamic flow	Electric current
Force acting on cylinder	Force acting on conductor
Magnus effect	Hall effect

Slides for lectures 4 - 5 are available at

http://purple.univ-littoral.fr/$\sim$boris/KyotoLect4.pdf