Derivation of the Nonlinear Normal Modes from the Reduced Problem

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Many thanks
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This work has benifited much from my earlier research with *John Delos* on normal forms near periodic orbits

Many aspects of this work are greatly influenced by the collaboration with *Boris Zhilinskiĭ*

Central ideas of the talk:

- Testing the limits of the approximate integrability, i.e., of KAM dynamics
- Dynamical structure on the initial and reduced phase space (RPS)
- in particular: Periodic orbits vs the stationary points on the RPS
- nonlinear normal modes [*J. Montaldi, M. Roberts, and I. Stewart*], the basic set of periodic orbits
- relation to quantum systems: the limits of the EBK quantization and the periodic orbit theory

Reduced problem:

- principal approximate integral n
- reduced phase space RPS, its topology
- coordinates θ on the RPS
- invariant polynomials and symmetries

Normal form near a stable equilibrium:

- the formal series parameter vs n
- energy E as a function on the RPS
- stationary points of NF, i.e, of $E(\theta; n)$
- the simplest Morse function on the RPS

Nonlinear normal modes:

- relation to the stationary points of NF
- action and period as functions of energy
- the orbit in the phase space $q(t), p(t)$

Example dynamical systems

Cubic oscillator

$$H = \frac{1}{2}(q^2 + p^2) + \varepsilon\frac{1}{3}q^3$$

Hénon-Heiles problem

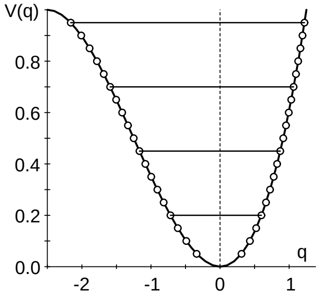
$$H = \frac{1}{2}(p_x^2 + q_x^2) + \frac{1}{2}(p_y^2 + q_y^2) + \varepsilon\left(\frac{1}{3}q_x^3 - q_x q_y^2\right)$$

Hydrogen atom in crossed fields

$$\begin{aligned} H &= \frac{1}{2}(p_b^2 + p_e^2 + p_p^2) - \frac{1}{r} \\ &\quad + Fq_e \\ &\quad + \frac{1}{2}GL_b + \frac{1}{8}G^2(q_e^2 + q_p^2) \end{aligned}$$

Potential energy of the 1D cubic oscillator

(lines give energies at which orbits are calculated)



coordinate q

Normal Form transformation

a near identity change of coordinates

$$x \to y + \epsilon y^2 + \ldots$$

which attempts to make the o.d.e. linear

$$\frac{dx}{dt} = Ax + v(x) \xrightarrow{??} \frac{dy}{dt} = Ay$$

but can only "simplify" the initial field $u(x)$

$$\frac{dx}{dt} = Ax + u(x) \longrightarrow \frac{dy}{dt} = Ay + v(y)$$

The Lie equation

consider infinitesimal change of variables

$$x = y + \epsilon w(y), \quad w(y) \sim O(|y|^2).$$

then the transformed o.d.e. is

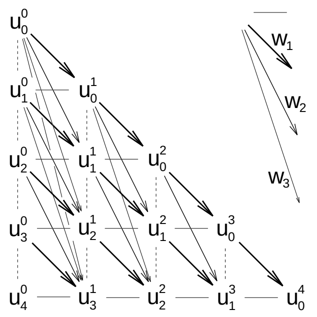
$$\begin{aligned} \frac{dy}{dt} &= Ay + \epsilon Aw(y) - \frac{dw}{dy}\epsilon\dot{y} + \epsilon u(y + \epsilon w(y)) \\ &= Ay + \epsilon\left[Aw(y) - \frac{dw}{dy}Ay\right] + \epsilon u(y) + \ldots, \\ &= Ay + \mathfrak{L}_A w(y) + \epsilon u(y) + \ldots \end{aligned}$$

we now try to solve the Lie equation

$$\mathfrak{L}_A w(y) + \epsilon u(y) \stackrel{?}{=} 0$$

and find $w(y)$, the *generator* of the normal form

Lie exponent $\exp(\epsilon\mathfrak{L}_W)$
and accumulation of terms



Lie triangle

Normal form of the 1D cubic oscillator

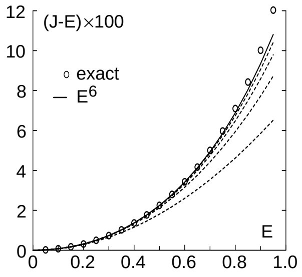
Energy as function of action $E = H_{\text{n.f.}}(J)$

$$J - \frac{5}{12}\epsilon^2 J^2 - \frac{235}{432}\epsilon^4 J^3 - \frac{38\,585}{31\,104}\epsilon^6 J^4 - \frac{2\,663\,129}{746\,496}\epsilon^8 J^5 - \frac{156\,934\,505}{13\,436\,928}\epsilon^{10} J^6 + \ldots$$

action as function of energy $n(E)$

$$E + \frac{5}{12}\epsilon^2 E^2 + \frac{385}{432}\epsilon^4 E^3 + \frac{85\,085}{31\,104}\epsilon^6 E^4 + \frac{7\,436\,429}{746\,496}\epsilon^8 E^5 + \frac{1\,078\,282\,205}{26\,873\,856}\epsilon^{10} E^6 + \ldots$$

Action $J(E)$ in the 1D cubic oscillator calculated from the ϵ^{10} normal form



energy in units relative to the barrier

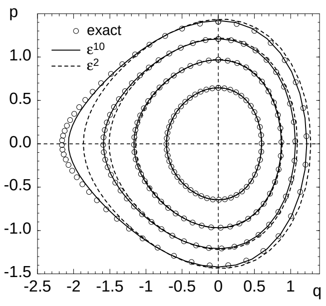
dashed lines give lower order approximations

Transformation of coordinates

$$z = q + ip \to z + \ldots = \exp(\epsilon\mathfrak{L}_W)(z)$$

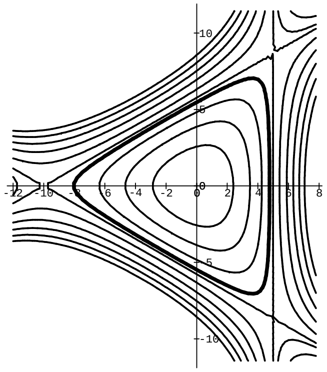
$$\begin{aligned} z &+ \epsilon\left(\frac{z^2}{4} - \frac{z\bar z}{2} - \frac{\bar z^2}{12}\right) \\ &+ \frac{\epsilon^2}{2}\left(\frac{z^3}{12} - \frac{z^2\bar z}{9} + \frac{5z\bar z^2}{12} - \frac{\bar z^3}{24}\right) \\ &+ \frac{\epsilon^3}{3!}\left(\frac{5z^4}{144} + \frac{61z^3\bar z}{48} - \frac{5z^2\bar z^2}{2} - \frac{7z\bar z^3}{48} - \frac{\bar z^4}{48}\right) \end{aligned}$$

Trajectories of the 1D cubic oscillator calculated from the ϵ^6 and ϵ^{10} normal form



coordinate

Hénon-Heiles potential $V(x,y)$
calculated for $\epsilon = 0.1$ and
$E/E_{\text{saddle}} = 0.2, 0.45, 0.7, \mathbf{0.9}, 1, 1.2, \ldots$



Integrity basis for the Hénon-Heiles problem

reduced phase space RPS $= CP_1 \sim S_2$

angular momentum representation for two oscillators $z_\alpha = q_\alpha + ip_\alpha$

$$\mathbf{J} = \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} z_x\bar z_y + z_y\bar z_x \\ iz_x\bar z_y - iz_y\bar z_x \\ z_x\bar z_x - z_y\bar z_y \end{pmatrix}$$

generating function

$$g(E \oplus A_2 \to A_1; \lambda) = \frac{1}{(1-\lambda^2)(1-\lambda^3)}$$

invariant polynomials

$$\begin{aligned} n &= 2J = \frac{1}{2}\rho^2 = \frac{1}{2}(\bar z_x z_x + \bar z_y z_y) \\ \mu &= J_2^2 \\ \xi &= \frac{1}{2}J_3(3J_1^2 - J_3^2) \end{aligned}$$

scaled polynomial coordinates on RPS $= S_2$

$$\begin{aligned} |\mathbf{J}| \to Jn^{-1} &= \frac{1}{2} \\ \mu \to \mu n^{-2} &= \cos(\theta)^2/4 \\ \xi \to \xi n^{-3} &= -\sin(\theta)^3\cos(3\varphi)/16 \end{aligned}$$

Normal form for the Hénon-Heiles problem

$$\begin{aligned} n &- \epsilon^2 n^2\left(\frac{5}{12} - \frac{7}{3}\mu\right) \\ &- \epsilon^4 n^3\left(\frac{67}{432} + \frac{7}{36}\mu - \frac{56}{9}\xi\right) + \ldots \end{aligned}$$

	n	$\epsilon^2 n^2$	$\epsilon^4 n^3$	$\epsilon^6 n^4$	$\epsilon^8 n^5$
1	1	$-\frac{5}{12}$	$-\frac{67}{432}$	$-\frac{42229}{155520}$	$-\frac{15624833}{18662400}$
μ		$\frac{7}{3}$	$-\frac{7}{36}$	$-\frac{76447}{6480}$	$-\frac{11656729}{2332800}$
ξ			$\frac{56}{9}$	$\frac{2093}{135}$	$\frac{353843}{8100}$
μ^2				$\frac{115171}{1944}$	$\frac{2217943}{233280}$
$\mu\xi$					$\frac{6701639}{4050}$

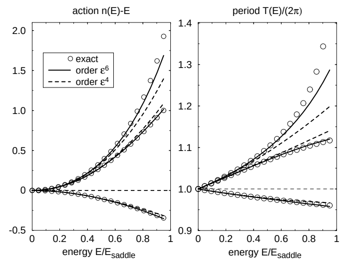
Stationary points of the normal form
of the Hénon-Heiles problem

point	$\mathfrak{g}$	ξ	μ	θ	φ
$\Pi_{7,8}$	C_3	0	1/4	0	–
$\Pi_{1,2,3}$	C_2	$-1/16$	0	$\pi/2$	$0, \pm 2\pi/3$
$\Pi_{4,5,6}$	C_2'	1/16	0	$\pi/2$	$\pi, \pm\pi/3$

energy at the stationary points

$\mathfrak{g}$	energy in units of n with $\tau = n^2\epsilon$
C_3	$1 + \frac{1}{6}\tau - \frac{11}{54}\tau^2 + \frac{1171}{2430}\tau^3 - \frac{217567}{145800}\tau^4$
C_2	$1 - \frac{5}{12}\tau - \frac{235}{432}\tau^2 - \frac{38585}{31104}\tau^3 - \frac{2663129}{746496}\tau^4$
C_2'	$1 - \frac{5}{12}\tau + \frac{101}{432}\tau^2 + \frac{108467}{155520}\tau^3 + \frac{35328559}{18662400}\tau^4$

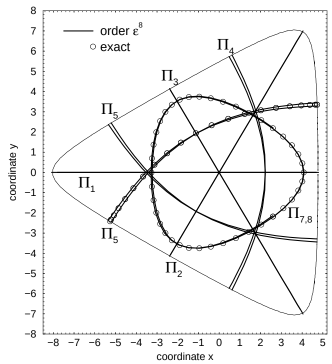
Action $n(E)$ and period $T(E)$
in the Hénon-Heiles problem
at three stationary points



energy in units relative to the barrier

solid and dashed lines show ε^6 and ε^4 order
normal form approximations

Nonlinear normal modes
of the Hénon-Heiles oscillator
calculated from the ϵ^8 and normal form



energy at 90% of E_{saddle}

Rydberg atom in crossed fields

Hamilton function in 3D

$$H = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r} + Fq_e + \frac{1}{2}GL_b + \frac{1}{8}G^2(q_e^2 + q_p^2)$$

Hamilton function in the KS space

$$\begin{aligned} H &= \frac{1}{2}(p^2 + q^2) \\ &\quad + \frac{1}{3}f(q_2q_1 - q_4q_3)q^2 + \frac{1}{2}g(q_2p_3 - q_3p_2)q^2 \\ &\quad + \frac{1}{8}g^2(q_2^2 + q_3^2)(q_1^2 + q_4^2)q^2 \\ &= \frac{4}{\omega}, \end{aligned}$$

Kustaanheimo-Stiefel transformation:

$$M_{\text{KS}} = \begin{pmatrix} q_1 & -q_2 & -q_3 & q_4 \\ q_2 & q_1 & -q_4 & -q_3 \\ q_3 & q_4 & q_1 & q_2 \\ q_4 & -q_3 & q_2 & -q_1 \end{pmatrix},$$

coordinates in the initial 3D space:

$$(q_b, q_e, q_p, 0)^T = M_{\text{KS}}(q_1, q_2, q_3, q_4)^T$$

$$r = q_b^2 + q_e^2 + q_p^2 = q_1^2 + q_2^2 + q_3^2 + q_4^2$$

momenta in the initial 3D space:

$$(p_b, p_e, p_p, \Pi)^T = \frac{1}{2r}M_{\text{KS}}(p_1, p_2, p_3, p_4)^T$$

where we impose an extra restriction

$$\zeta \stackrel{\text{def}}{=} -2r\Pi = q1p4 - q4p1 + q3p2 - q2p3 = 0$$

Parameterization

scaled field parameters:

$$g = 4G/\omega^2, \quad f = 24F/\omega^3, \quad \omega = \sqrt{-8E}$$

uniform field parameters:

$$F = \frac{1}{3}F_sS\left(\frac{\omega}{2}4\right)^4, \quad G = G_sS\left(\frac{\omega}{2}4\right)^3$$

formal smallness parameters:

$$S = \left(\frac{2}{\omega}\right)^2\sqrt{G^2 + \left(3F\frac{2}{\omega}\right)^2} \approx \tau = n\frac{\omega}{2}S$$

How small should the smallness be?

Stark ionization along

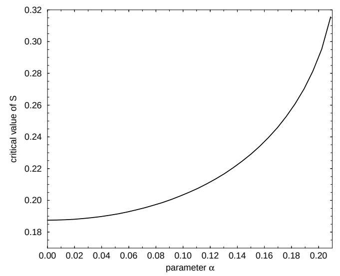
$$q1 = q2 = 0, \quad q3 = q4 = \sqrt{\rho/2}$$

$$\frac{1}{2}\rho - \frac{1}{6}\rho^2F_s\left(S\frac{\omega}{2}\right) + 1/32\rho^3G_s^2\left(S\frac{\omega}{2}\right)^2 = \frac{4}{\omega}$$

the critical smallness parameter S_{saddle}

$$\begin{aligned} S &= -\frac{(4F_s - \sqrt{43F_s^2 - 27})}{729(F_s - 1)^2(F_s + 1)^2} \\ &\quad \times (70F_s^2 - 4F_s\sqrt{43F_s^2 - 27} - 54) \\ &\approx \tfrac{3}{16} - \tfrac{75}{256}(F_s - 1) + \tfrac{9579}{16384}(F_s - 1)^2 \ldots \end{aligned}$$

Critical value of the uniform field amplitude
of the orthogonal electric and magnetic
fields



The principal integral of motion

$$2n = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2 + p_4^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2)$$

The KS restriction integral

$$\zeta = q_1 p_4 - q_4 p_1 + q_3 p_2 - q_2 p_3 = 0$$

The reduced phase space Σ with $n =$ const
can be described by two vectors:
the angular momentum $\mathbf{L}$ and
the Runge-Lentz vector $\mathbf{K}$

$$n^2 = \mathbf{K}^2 + \mathbf{L}^2, \quad \mathbf{K} \cdot \mathbf{L} = 0$$

the topology of the RPS is $S_2 \times S_2$

The reduced phase space

$$\Sigma = S_2 \times S_2, \quad n = \text{const}$$

$$n^2 = \mathbf{K}^2 + \mathbf{L}^2, \quad \mathbf{K} \cdot \mathbf{L} = 0$$

The angular momentum componenets:

$$\begin{aligned} L_b &= q_2 p_3 - q_3 p_2 \\ L_e &= \frac{1}{2}(q_2 p_4 + q_3 p_1 - q_1 p_3 - q_4 p_2) \\ L_p &= \frac{1}{2}(q_1 p_2 + q_3 p_4 - q_2 p_1 - q_4 p_3) \end{aligned}$$

The Runge-Lentz vector componenets:

$$\begin{aligned} K_b &= \frac{1}{4}(p_2^2 + q_2^2 + q_3^2 + p_3^2 - q_1^2 - p_1^2 - q_4^2 - p_4^2) \\ K_e &= \frac{1}{2}(p_4 p_3 - q_2 q_1 - p_1 p_2 + q_4 q_3) \\ K_p &= -\frac{1}{2}(q_3 q_1 + p_1 p_3 + q_4 q_2 + p_4 p_2) \end{aligned}$$

symmetry group G_4

$$\{I, \sigma, T_s = T \circ \sigma', T_2 = T \circ C_2\}$$

σ	reflection in the ep plane
T	time reversal
C_2	rotation around axis e
σ'	reflection in the eb plane

generating function

$$g(\{K, L\} \to G_4; \lambda) = \frac{1 + 2\lambda^2 + \lambda^4}{(1-\lambda)^2(1-\lambda^2)^2}$$

integrity basis polynomials

main: $K_e, \quad L_b, \quad K_b^2, \quad L_e^2$

auxiliary: $K_b L_e, \quad K_p^2 - L_p^2, \quad K_b L_e (K_p^2 - L_p^2)$

Normal form for the
Rydberg electron in crossed fields

$$\begin{aligned} Hnf &= 4/\omega \\ Hnf &= 2n\left(1 + \frac{\tau}{2}H_1 + \frac{\tau^2}{4}H_2 + \frac{\tau^3}{32}H_3 + \ldots\right) \end{aligned}$$

$$\begin{aligned} H_1 &= G_s L_b - F_s K_e \\ H_2 &= -(3L_b^2 - 2K_e^2 + 3K_b^2 + 2L_e^2 + 2(L_p^2 - K_p^2) - 3)G_s^2/4 \\ &\quad +(7K_e L_b - L_e K_b)F_s G_s/3 \\ &\quad -(51K_e^2 - 9L_e^2 + 17)F_s^2/36 \\ H_3 &= L_b(6L_e^2 - 6K_e^2 + 9K_b^2 + 5L_b^2 + 6(L_p^2 - K_p^2) - 9)G_s^3 \\ &\quad -K_e(125 - 43L_e^2 + 125K_e^2)F_s^3/9 \\ &\quad +(-45K_e L_e^2 + 45K_e^3 - 65K_e K_b^2 + 117K_e \\ &\quad -147L_b^2 K_e - 45K_e(L_p^2 - K_p^2) + 48L_b(L_e K_b))F_s G_s^2/6 \\ &\quad -(3K_b^2 L_b + 87L_e^2 L_b - 615K_e^2 L_b - 3L_b^3 - 232L_b \\ &\quad -3L_b(L_p^2 - K_p^2) + 108K_e(L_e K_b))F_s^2 G_s/18 \end{aligned}$$

Stationary points on $S_2 \times S_2$
action n and period Δ

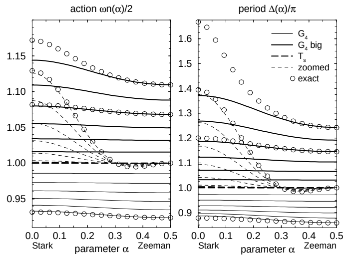
symmetry G_4

K_e	$\mp F_s - \tau F_s G_s^2(4G_s^2 - 3)/12$
	$\mp \tau^2 F_s G_s^2(-103G_s^2 - 60G_s^4 + 40G_s^6 + 21)/144$
L_b	$\pm G_s - \tau G_s F_s^2(4G_s^2 - 3)/12$
	$\mp \tau^2 G_s F_s^2(80G_s^6 - 104G_s^4 - 230G_s^2 + 51)/288$
$\frac{1}{2}n\omega$	$1 \mp S/2 + S^2(G_s^4/4 - 5G_s^2/3 + 35/12)/3$
Δ/π	$1 \mp S + S^2(G_s^4/4 - 5G_s^2/3 + 35/12)$

symmetry T_s

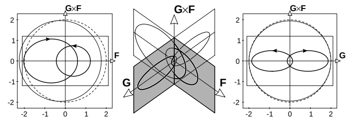
K_e	$-\tau F_s G_s^2(4G_s^2 + 1)/12$
L_b	$-\tau G_s F_s^2(4G_s^2 + 1)/12$
K_b	$\pm G_s$
	$\mp \tau^2 G_s F_s^2(80G_s^6 + 72G_s^4 - 146G_s^2 + 13)/288$
L_e	$\mp F_s$
	$\mp \tau^2 F_s G_s^2(6 - 77G_s^2 + 28G_s^4 + 40G_s^6)/144$
$\frac{1}{2}n\omega$	$1 - S^2 F_s^2(3G_s^2 - 2)/36$
Δ/π	$1 - S^2 F_s^2(3G_s^2 - 2)/12$

Action and period
of the nonlinear normal modes
of the Rydberg electron in the crossed fields



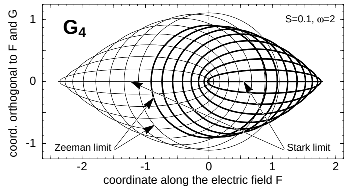
normal form prediction of order ϵ^3
for $S = 0.03, 0.06, 0.09, 0.12, 0.15, 0.18$

3D image of the nonlinear normal modes
of the Rydberg electron in crossed fields
$S = 0.2$ and $F_s = \cos(\pi/3)$ $(\alpha = 1/3)$



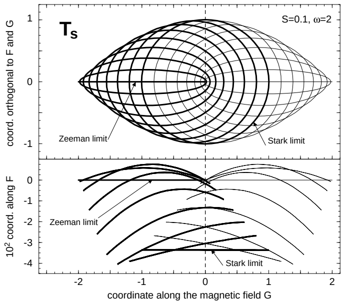
normal form prediction of order ϵ^3

nonlinear normal modes of symmetry G_4
of the Rydberg electron in crossed fields
$S = 0.1$ and $F_s = \cos(\alpha\pi)$ with $\alpha = 0 \ldots 1/2$



normal form prediction of order ϵ^3

nonlinear normal modes of symmetry T_s
of the Rydberg electron in crossed fields
$S = 0.1$ and $F_s = \cos(\alpha\pi)$ with $\alpha = 0 \ldots 1/2$



normal form prediction of order ϵ^3