Organization of sequences of bifurcations of periodic orbits

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Numerical calculations have shown that bifurcations of periodic orbits of Hamiltonian systems often occur in organized groups. Normal-form theory provides an explanation.

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The study of periodic orbits (PO's) provides a route to understanding nonintegrable dynamics [1], and periodic orbits have also become central in the study of "quantum chaos" [2]. Classical PO's produce clearly-observable effects in atomic spectra: in the case of near-zero-energy states of an atom in magnetic or electric fields, the large-scale structure of the absorption spectrum is formed by those short-time orbits of the electron that begin at and return to the nucleus. Many such recurrences have been identified in experimental data [3,4].

If fixed parameters of the system, such as energy or field strength, are made to vary, then the family of periodic orbits changes. For instance, as energy increases new periodic orbits can be created, and as order changes to chaos this results in a proliferation of periodic orbits. In atoms in fields the new orbits are observed as new peaks that emerge in the recurrence spectrum when the scaled energy changes [5]. It follows that the mechanism of individual bifurcations of periodic orbits, and their patterns and sequences (their "organization"), are of fundamental importance to the study of dynamics of both classical and quantum systems.

The standard way to study bifurcations of periodic orbits is to numerically generate trajectories in the neighborhood of a given periodic orbit, and to plot a series of Poincaré surfaces of section (SOS) for different values of the parameter. The SOS method does not, however, address the question of *why* certain phenomena occur; it gives only "experimental data", i.e., results of purely numerical experiments. Using this method Shaw [6, a] and Mao and Delos [6, b] discovered organized sequences of bifurcations associated with the perpendicular orbit of the diamagnetic Kepler problem, discussed below.

In this paper we show how such organized sequences of bifurcations can be understood using "Normal Form Theory". This theory [1,7–9] is a form of classical perturbation theory which, when combined with the principles of bifurcation theory, gives a qualitative and a quantitative description of motion near a periodic orbit, and of how that motion changes when parameters of the system are made to vary.

We consider the diamagnetic Kepler problem, one electron in highly-excited states of energy E in a hydrogen atom in a magnetic field B, with Hamiltonian

$$H = \frac{p_\rho^2 + p_z^2 - (\rho^2+z^2)^{-1/2} + \rho^2}{2} = EB^{-2/3} = \varepsilon, \quad (1)$$

where ε is the "scaled energy" [6]. This system has an orbit lying in the $z=0$ plane, perpendicular to the applied magnetic field. In rotated semiparabolic coordinates (λ,σ), such that $(\rho,z) = (\lambda^2-\sigma^2,\ 2\lambda\sigma)$, this perpendicular orbit lies on the λ axis, and Hamiltonian (1)

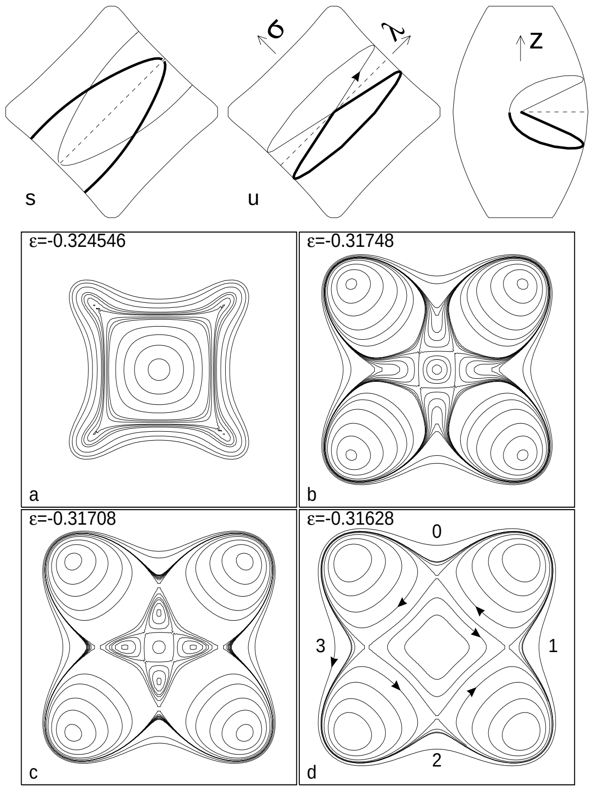

FIG. 1. (above) Periodic trajectories created near the perpendicular orbit after it undergoes a period-4 bifurcation: (s) stable trajectories in (λ,σ) coordinates; (u) unstable trajectories in (λ,σ) and in (ρ,z) coordinates for scaled energy $\varepsilon = -0.31$. (below) "Organized" sequence of bifurcations near the period-4 bifurcation of perpendicular orbit.

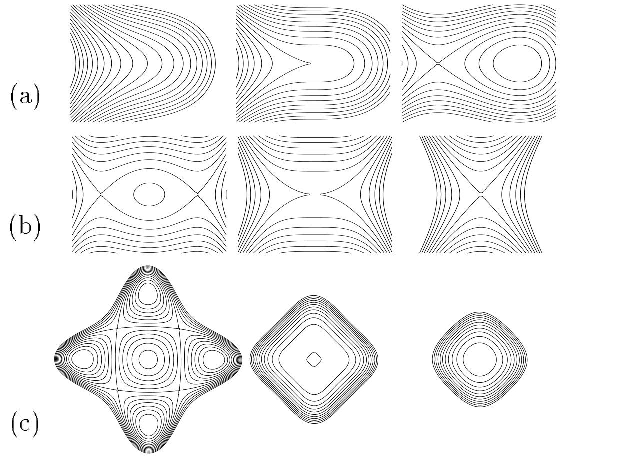

FIG. 2. Generic bifurcations discussed in this paper: (a) saddle-node (no symmetry), (b) pitchfork (symmetry C_2), and (c) period-4 (symmetry C_4, case of weak resonance). Curves correspond both to contours of normal forms $\mathcal{H}$ and to interpolated Poincaré surfaces of section.

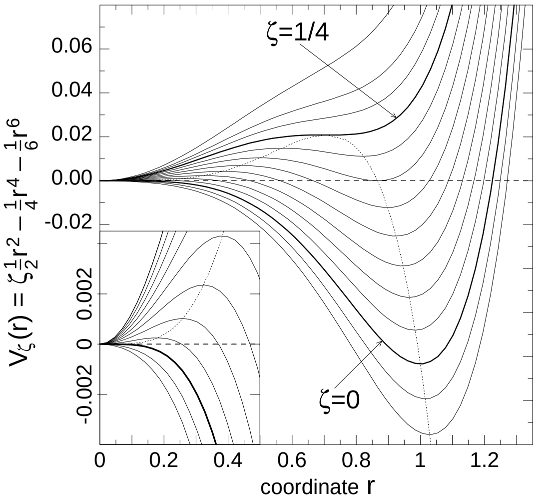

FIG. 3. "Organized" one-parameter bifurcations of stationary points for the function $V_\zeta(r)$ (3b). Dotted line gives the position of stationary points.

becomes

$$H_\varepsilon(p_\lambda, p_\sigma, \lambda, \sigma) = \tfrac{1}{2}(p_\lambda^2 + p_\sigma^2) + 4(\lambda^2+\sigma^2)\left[\tfrac{1}{2}(\lambda^2-\sigma^2)^2 - \varepsilon\right] = 2. \quad (2)$$

It contains one parameter, the scaled energy ε.

An observation. Shaw, Mao and Delos (SMD) [6] were studying the period-4 bifurcation of the perpendicular orbit, and the role this bifurcation plays in the creation of the new PO that they call "pac-man" [Fig. 1(u)]. A general theorem by Meyer [7] describes generic bifurcations of PO's (Fig. 2). In particular, it asserts that one of the two generic forms of a period-4 bifurcation has the "island-chain" pattern on the surface of section [Fig. 2(c)]: four stable PO's (O-points) and four unstable PO's (X-points) move toward the central stable PO, collide with it simultaneously, and disappear, leaving the central PO stable.

SMD saw this phenomenon in their calculations, but they saw that it was part of a more complicated sequence of events (Fig. 1): (a) A stable-unstable pair of period-4 orbits was created nearby in a "saddle-node" bifurcation. (b) A second such pair was created in a similar fashion. (c) The separatrices rearranged into two concentric 4-island-chains. (d) The actual period-4 bifurcation of Meyer's theorem occurred: the inner chain shrank and collapsed onto the perpendicular orbit leaving only the outer chain. All these observations came out of careful examination of many numerical calculations; at the time, SMD could neither anticipate nor explain such sequences of events. They observed, however, that most of the bifurcations of the perpendicular orbit occur through such ordered sequences. We have reasons to believe that these ordered sequences occur in many systems [10].

We use the term *organization* when we refer to such sequences of bifurcations. It is this kind of phenomenon for which we seek an explanation in Normal Form Theory.

An analogy. Consider for a moment a simpler one-parameter Hamiltonian,

$$\mathcal{H}_\zeta = p_r^2/2 + V_\zeta(r), \quad (3a)$$
$$V_\zeta(r) = \zeta r^2/2 - r^4/4 + r^6/6. \quad (3b)$$

In this case a bifurcation is any creation or destruction of fixed points of $\mathcal{H}_\zeta$, i.e., of points where $p_r = \partial V_\zeta/\partial r = 0$. In Fig. 3 we see that for ζ large and positive, there is a single stable fixed point at $r = 0$. As ζ decreases through $1/4$, a new local maximum and minimum of $V_\zeta(r)$ are created at a critical point at $r_0 = 1/\sqrt{2}$. The new unstable and stable fixed points of $\mathcal{H}_\zeta$ move apart as ζ continues to decrease. This is an example of a "saddle-node" bifurcation [Fig. 2(a)]. Due to the symmetry of (3) about $r = 0$ a twin stable-unstable pair appears at the same time at $r_0' = -1/\sqrt{2}$. As ζ decreases through zero, the twin unstable points move to the origin, "collide" with it, and disappear, leaving the origin unstable for negative ζ. This is an example of a "pitchfork" bifurcation [Fig. 2(b)].

Both bifurcations of Hamiltonian (3) follow well-known patterns. The saddle-node bifurcation (which is generic) has a locally-cubic structure $(r-r_0)^3 - \zeta(r-r_0)$ [Fig. 2(a)], while the pitchfork [generic under the restricting symmetry $V(-r) = V(r)$] has a locally-quartic structure $\zeta r^2 - r^4$ [Fig. 2(b)]. The special point about Hamiltonian (3) is that it contains these two bifurcations in an organized sequence caused by a monotonic change of single parameter.

Contour plots and the surface of section. Now consider another Hamiltonian function

$$\mathcal{H}_\varepsilon(I, \varphi) = (\varepsilon - \varepsilon_4)\alpha I + (\beta_1 + \gamma_1 \cos 4\varphi)I^2 + \beta_2 I^3 \quad (4)$$

with $\alpha = -1.26$, $\beta_1 = -0.11$, $\gamma_1 = 0.04$, $\beta_2 = 0.71$, $\varepsilon_4 \approx -0.3163$. The important properties of this function are: (i) the coefficient of I passes through zero at $\varepsilon = \varepsilon_4$ while the other coefficients remain nonzero; (ii) $|\gamma_1| < |\beta_1|$, the "case of weak resonance"; (iii) for any φ, the coefficients of I^2 and I^3 have opposite signs, $\mathrm{sgn}(\beta_1 \pm \gamma_1) = -\mathrm{sgn}(\beta_2)$.

Hamiltonian (4) has fixed points whenever $\partial H/\partial I = \partial H/\partial \varphi = 0$. To find them, note that they must lie on the lines $\varphi = n\pi/4, n = 0 \ldots 7$. Then note that for any fixed φ, the substitution $I = r^2$ makes the function (4) qualitatively equivalent to $V_\zeta(r)$ in Eq. (3b): as ε increases through ε_4, the function $\mathcal{H}_\varepsilon(I, \varphi)$ undergoes exactly the same sequence of bifurcations.

Finally, let us define $p_\sigma = \sqrt{2I}\cos\varphi$, $\sigma = \sqrt{2I}\sin\varphi$, and let us plot contours of (4) in the (σ, p_σ) plane. The result is Fig. 1. The contours of this simple function of two variables and one parameter are indistinguishable from the surface-of-section plots obtained laboriously by Shaw, Mao and Delos [6]. The organized sequence of bifurcations which they found but could not explain is entirely contained in this one function.

Normal-Form Theory is a systematic procedure for locally converting the original Hamiltonian (2) near a given PO into an effective Hamiltonian like (4). At its lowest level of approximation, retaining only linear and quadratic terms in I, it leads to the wonderful theorem of Meyer [7, a], which asserts that PO's typically bifurcate in just five ways, depending on the period-multiplication factor [11]. At higher levels of approximation (with more terms) this theory leads to effective Hamiltonians that can also describe bifurcations of other related orbits in the greater neighborhood of the original PO, and the sequential organization of these bifurcations.

In this work we report the first complete quantitative computer realization of the normal form expansion about a PO [12]. We have used it to calculate formulas like Eq. (4) and thereby to explain observations of bifurcations near the perpendicular orbit.

Let us see how normal-form theory works and what it contributes, by examining two other bifurcations of the perpendicular orbit: a period-3 bifurcation, which occurs when the dynamics is almost regular, and a period-1 bifurcation, which marks the onset of global chaos.

A period-3 bifurcation occurs at a very low value of ε ($\varepsilon_3 \approx -0.483$). Because of symmetries of the system [6,11,12], this bifurcation produces a six-island chain (Fig. 4). SMD guessed that this chain was the ultimate outcome of a sequence of events similar to the period-4 case in Fig. 1 (outer ring of saddle-nodes, inner ring, reconnection, and disappearance of inner island chain). A numerical search for this sequence of bifurcations was unsuccessful.

Our normal-form calculations [12] showed that the parameter β_1 increases as ε becomes more negative. Implementing the procedure near the 3-bifurcation, we obtained the normal form

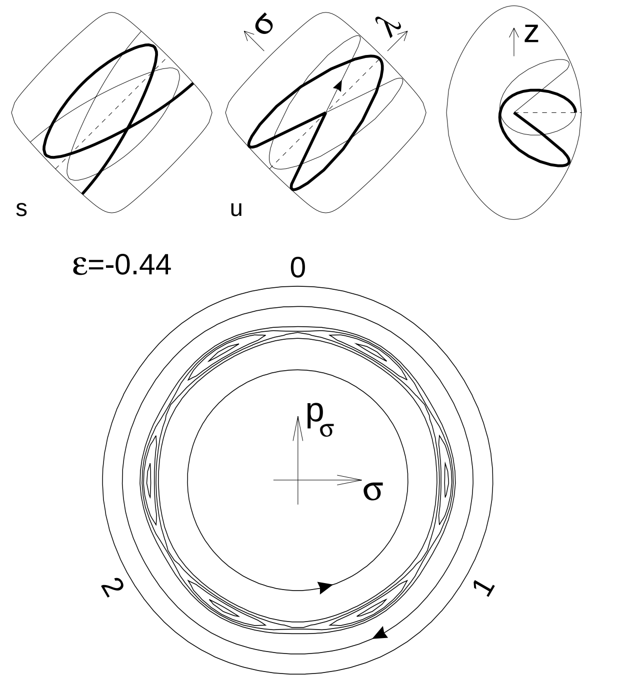

FIG. 4. (above) Periodic trajectories created in the period-3 bifurcation: (u) stable trajectories in (λ, σ) coordinates; (s) unstable trajectories in (λ, σ) and (ρ, z) coordinates. (below) Constant level sections of the 1:3 normal form.

$$\mathcal{H}_\varepsilon(I, \varphi) = (\varepsilon - \varepsilon_3)\alpha I + \beta_1 I^2 + (\beta_2 + \gamma_2 \cos 6\varphi) I^3 \quad (5)$$

with $\alpha = -0.80$, $\beta_1 = 0.04$, $\beta_2 = 0.35$, $\gamma_2 = 0.005$, $\varepsilon_3 \approx -0.483$. (The symmetry of the system eliminates the $I^{3/2}\cos 3\varphi$ term, leaving $I^3 \cos 6\varphi$ as the main resonance term [11,12].) Here the important point is that the quadratic and cubic terms have the same sign. Therefore the saddle-node bifurcations and the organized sequence are not present. The new island chain is born directly out of the central PO, and steadily moves away as ε increases.

A period-1 pitchfork bifurcation occurs at $\varepsilon = \varepsilon_1 \approx -0.127$. It is the last bifurcation of the perpendicular orbit: here this orbit goes unstable, and as ε increases further, chaos dominates the phase space. Does this bifurcation happen in the same way as period-4? By close examination of numerical Poincaré surfaces of section, we found a saddle-node bifurcation that created a distant small island of order in the sea of chaos. Our calculations [12] show that the normal form is

$$\mathcal{H}_\varepsilon(p_\sigma, \sigma) = \tfrac{1}{2}p_\sigma^2 + \tfrac{1}{2}k_1(\varepsilon - \varepsilon_1)\sigma^2 + k_2\sigma^4 + k_3\sigma^6 \quad (6)$$

with $k_1 = -6.84$, $k_2 = -0.46$, $k_3 = 0.40$. Again the important fact is that the σ^4 and σ^6 terms have opposite signs. The consequence is again an organized sequence, this time exactly like that for Hamiltonian (3): saddle-node bifurcations occur on the $\pm\sigma$ axes; then the newly-created unstable points move toward the origin, collide with it, and leave it unstable (Fig. 5).

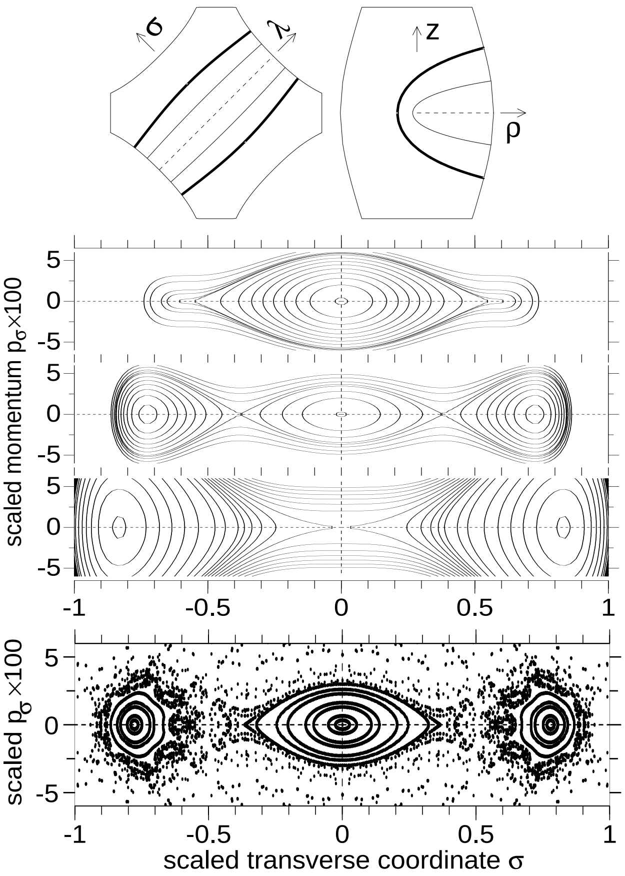

FIG. 5. (above) Periodic trajectories created near the perpendicular orbit before it undergoes its last bifurcation (the period-1 pitchfork) and becomes unstable. Orbits are drawn in (λ, σ) and (ρ, z) coordinates for scaled energy $\varepsilon = -0.14$. Stable trajectory is shown by a bold line. (center) Contour plot of the normal form $\mathcal{H}_\varepsilon(\sigma, p_\sigma)$ near this bifurcation: (descending) $\varepsilon - \varepsilon_1 = -0.022$, -0.015, 0.002. (below) Numerical Poincaré surface of section near this bifurcation: $\varepsilon = -0.14$, $\varepsilon - \varepsilon_1 = -0.01273$.

Comparison with the numerically-generated SOS (Fig. 5) displays another important property of the Normal-Form method: the theory converts the true Hamiltonian, for which the dynamics is almost entirely chaotic, into a different effective Hamiltonian, which has only integrable dynamics. In so doing, it preserves locally the essential short-time dynamics, i.e., it preserves the structure and stability of nearby short periodic orbits. Normal Form theory replaces the complicated exact dynamics seen in the SOS by contour plots of simple functions, and thereby it provides simple explanations of the organization of sequences of bifurcations in the vicinity of a periodic orbit.

How general is this phenomenon of organization of bifurcations [10]? Let us suppose that the normal form, generally a divergent series, nevertheless provides an adequate representation of the dynamics in the local area of interest around a periodic orbit. The principal order of the phenomenon is r^6 (I^3) and therefore it will suffice if, for some distance from the central orbit r_0, terms r^8 and r^{10} do not result in qualitative changes within $r < r_0$. If this holds, the organization we have described occurs whenever terms r^4 and r^6 have opposite signs. Therefore these organized sequences of bifurcations must be common.

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