On a three degree of freedom generalization of the two degree of freedom Hénon-Heiles Hamiltonian

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Abstract. We study a class of three degrees of freedom (3-DOF) Hamiltonian systems that share certain characteristics with the 2-DOF Hénon-Heiles system. Our systems represent a 1:1:1 resonant three-oscillator whose principal nonlinear perturbation is the cubic potential term xyz with tetrahedral symmetry. After normalizing and reducing the 1:1:1 oscillator symmetry, we show that near the limit of linearization all our systems can be described as a one-parametric family. Such reduced systems have been suggested earlier by Hecht in 1960 and later by Patterson in 1985 to model triply degenerate vibrations of tetrahedral molecules. We describe relative equilibria of these systems, classify all qualitatively different family members, and discuss bifurcations of relative equilibria involved in the transitions from one region of regular parameter values to the other.

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1. Introduction

In [1] M. Hénon and C. Heiles introduced their now famous model Hamiltonian

$$H(r, \theta, z, p_r, p_\theta, p_z) = \tfrac{1}{2}(p_r^2 + p_z^2) + \frac{p_\theta^2}{2r^2} + \tfrac{1}{2}(r^2 + z^2) + r^2 z - \tfrac{1}{3}z^3, \qquad (1)$$

where (r, θ, z) are cylindrical coordinates in $\mathbf{R}^3$, in order to study the existence of a third integral of motion (besides the energy E and the z-component of the angular momentum L_z) in axially symmetric galaxies. The system with Hamiltonian (1) has three degrees of freedom (3-DOF). It has been studied in [2, 3, 4] where it is called the 3D Hénon-Heiles Hamiltonian.

After reducing the axial symmetry of (1), the system can be represented by a family of the 2-DOF Hamiltonians parameterized by the value of the z-component of the angular momentum $L_z \equiv p_\theta$. Hénon and Heiles studied the particular case $p_\theta = 0$. They also formally extended the coordinates (r, z) with $r \geq 0$ to the 2-plane $\mathbf{R}^2$. The resulting Hamiltonian

$$H(x, y, p_x, p_y) = \tfrac{1}{2}(x^2 + p_x^2) + \tfrac{1}{2}(y^2 + p_y^2) + \epsilon\big(x^2 y - \tfrac{1}{3}y^3\big), \qquad (2)$$

where we replaced $(r, z) \to (x, y)$ in order to indicate the above extention and introduced ϵ to scale the cubic perturbation potential, is since known as the 2-DOF (or 2D) Hénon-Heiles Hamiltonian. The specific form of its potential was chosen “because:

(i) it is analytically simple; ...(ii) at the same time, it is sufficiently complicated to give trajectories far from trivial" [1].

As Hénon and Heiles found numerically in [1], the additonal integral of motion did not exist in the 2-DOF system with Hamiltonian (2). Their study resulted in first illustrations of Hamiltonian chaos, and their system has since been studied extensively both numerically and analytically, see [5] for a detailed list of references. It has served not only as a model for the dynamics near the center of a galaxy but also in molecular physics where it has been used to describe doubly degenerate vibrations of molecules whose equilibrium configuration has one or several three-fold symmetry axis [6], such as H_3^+ [7], P_4 or CH_4, and SF_6.

The Hamiltonian (2) is symmetric under the action of the finite group $D_3 \times \mathcal{T}$. Here D_3 is the dihedral symmetry group of the equilateral triangle, and $\mathcal{T} = \{1, T\}$ is the *time reversal* or *momentum reversal* group generated by $\mathcal{T} : (x, y, z, p_x, p_y, p_z) \to (x, y, z, -p_x, -p_y, -p_z)$. Although the finite symmetry of (2) was not particularly emphasized in [1], it has one very important consequence, namely the *a priori* existence at low energy of eight families of periodic orbits called *nonlinear normal modes* [8, 9, 10]. In fact, this is a property of *any* $D_3 \times \mathcal{T}$ symmetric 1:1 resonant 2-oscillator.

1.1. Generalized Hénon–Heiles Hamiltonian

In this work we propose and study the following three-dimensional generalization of the Hamiltonian (2)

$$H_\epsilon(x, y, z, p_x, p_y, p_z) = H_0 + V_\epsilon(x, y, z), \tag{3a}$$

where x, y, z are Cartesian coordinates in $\mathbf{R}^3$, p_x, p_y, p_z are the corresponding conjugate momenta. We assume that (x, y, z) transform according to the *vector representation* of the orthogonl group $\mathrm{O}(3)$ of transformations of $\mathbf{R}^3$. The zero-order Hamiltonian in (3a)

$$H_0(x, y, z, p_x, p_y, p_z) = \tfrac{1}{2}(p_x^2 + p_y^2 + p_z^2) + \tfrac{1}{2}(x^2 + y^2 + z^2) \tag{3b}$$

represents three harmonic oscillators with equal frequencies, that is, the 1:1:1 resonant (isotropic) harmonic oscillator. The perturbation potential

$$V_\epsilon(x, y, z) = \epsilon K_3\, xyz + \epsilon^2 K_0 (x^2 + y^2 + z^2)^2 + \epsilon^2 K_4 (x^4 + y^4 + z^4) \tag{3c}$$

is a polynomial in (x, y, z) of degree 4. The dimensionless smallness parameter ϵ characterizes the magnitude of the perturbation, while parameters K_0, K_3 and K_4 give the relative strength of each perturbation term. We assume that these parameters are of the order of 1. Note that we use K_3 in order to keep track of the contribution of the cubic potential term; in principle, this parameter can be absorbed into ϵ.

Comparison of the family (3) to the 2D Hénon-Heiles Hamiltonian (2).

(i) Like the 2D Hénon-Heiles system, the system with Hamiltonian (3) is probably not integrable. It is a genuinely three–dimensional system in the sense that the only exact first integrals are smooth functions of energy.

(ii) Both (2) and (3) have no compact Lie group of symmetries. Both are invariant under the action of a discrete group which includes rotations by $2\pi/3$ and the time reversal group $\mathcal{T}$ described above. Our Hamiltonian (3) is invariant under the action of the group $T_d \times \mathcal{T}$, where $T_d \subset \mathrm{O}(3)$ is the tetrahedral group of transformations of $\mathbf{R}^3$. This symmetry group has four conjugate subgroups $C_{3v} \times \mathcal{T}$ which are isomorphic to the symmetry group of (2).

(iii) Both (2) and (3) have principal cubic perturbation terms of the simplest possible analytic form which realize completely the respective point group symmetries D_3 and T_d. The polynomial V_ϵ in (3c) is the most general T_d-invariant polynomial in (x, y, z) of degree 4.

A complete proof of the nonintegrability of the system with Hamiltonian (3) is beyond the scope of this work and is not really necessary. However, direct computations for high degree $T_d \times \mathcal{T}$ invariant polynomials $F(x, y, z, p_x, p_y, p_z)$ show that $\{H_\epsilon, F\}$ does not vanish. Furthermore, numerical integration also reveals chaotic dynamics.

The symmetry properties of H_ϵ can be verified easily once the action of $O(3) \supset T_d$ is defined on the phase space $T\mathbf{R}^3$ using the diagonal extension of the standard action of $O(3) \supset T_d$ on $\mathbf{R}^3$ with coordinates (x, y, z), see [11, 12, 13] and Appendix A, to (p_x, p_y, p_z). The action and structure of $T_d \times \mathcal{T}$ is described in detail in [14].

The ring of polynomial invariants of the T_d action on $\mathbf{R}^3$ with coordinates (x, y, z) is generated freely three principal invariants

$$\mu_2 = x^2 + y^2 + z^2, \quad \mu_3 = xyz, \quad \text{and} \quad \mu_4 = x^4 + y^4 + z^4 \ .$$

This can be confirmed by a direct computation of the corresponding Molien generating function $g(\lambda) = 1/\big[(1 - \lambda^2)(1 - \lambda^3)(1 - \lambda^4)\big]$, with λ representing any of the variables (x, y, z), see for example [15, 16]. It follows that the potential V_ϵ in (3c) includes all principal invariants. It is the most general T_d-invariant polynomial in (x, y, z) of degree four. However, including only the cubic term μ_3 is sufficient to bring the symmetry of (3c) down to T_d.

Remark 1. *The most obvious physical realization of a system with Hamiltonian (3) is an atom (a spherical particle) trapped in a tetrahedral potential well. Hamiltonian (3) can be also used to model triply degenerate vibrational modes tetrahedral of the molecules A_4 or AB_4 [14]. In this latter case, however, we should include the term representing the squared length of the vibrational angular momentum $[(x, y, z) \times (p_x, p_y, p_z)]^2$. We will see later that omission of such term in (3) results in no loss of generality because Hamiltonian (3) with fourth degree terms in (3c) included has already enough parameters to represent qualitatively the above molecular systems.*

The natural starting point of the analysis of the family of systems with Hamiltonian (3) is near the linearization limit $\epsilon \to 0$. According to a theorem by Weinstein [17], a perturbed non-resonant k-oscillator near this limit has k families of short periodic orbits called *nonlinear normal modes*. In the presence of resonances, the oscillator can have more than k such families. The number and the properties of the modes depend primarily on the resonance and the symmetry of the perturbing nonlinear terms. Like in the case of the 2D Hénon-Heiles system [8, 9], nonlinear normal modes of our 3D system with Hamiltonian (3) are special periodic solutions characterized by a nontrivial *isotropy group* G, or *stabilizer*, which is a subgroup of the total symmetry group of the system $T_d \times \mathcal{T}$. Description of the subgroups G is given in Appendix A and in more detail in [13]. In [9] Montaldi, Roberts, and Stewart study special short period solutions for a system which near $\epsilon \to 0$ is equivalent to ours. Using their results we immediately obtain.

Theorem 1. *The system with Hamiltonian (3) has the minimum of* 27 *nonlinear normal modes which can be classified according to their stabilizers $G \subset T_d \times \mathcal{T}$ as follows.*

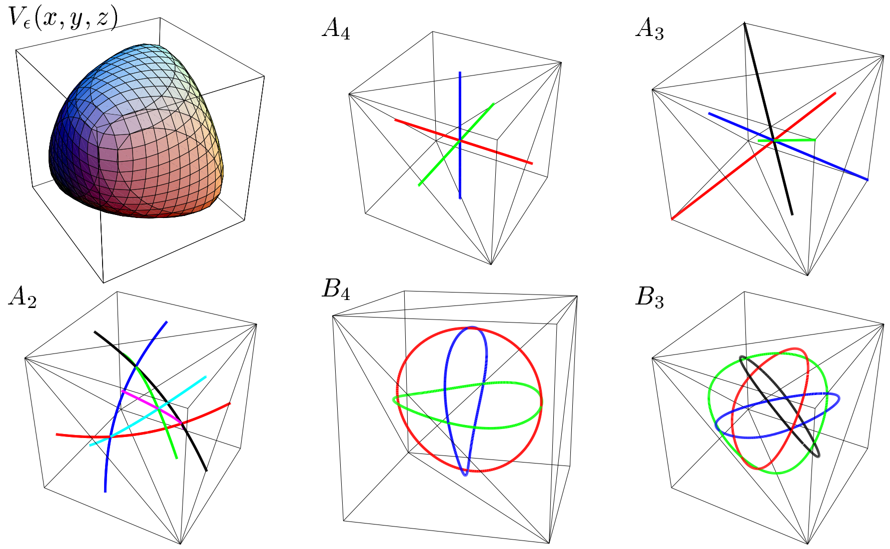

Figure 1. Equipotential surface of H_ϵ in (3) (top left). Configuration space representation of the nonlinear normal modes of the system with this Hamiltonian. These modes correspond to the relative equilibria of the normalized system which in turn correspond to the critical points of the $T_d \times \mathcal{T}$ group action on the reduced phase space $\mathbf{CP}^2$. The periodic orbits are computed for appropriate values of the parameters ϵ, K_3, K_4, and K_0 in (3).

cojugacy class of stabilizers	*shorthand notation*	*number of modes*
$D_{2d} \times \mathcal{T}$	A_4	3
$C_{3v} \times \mathcal{T}$	A_3	4
$C_{2v} \times \mathcal{T}$	A_2	6
$S_4 \times \mathcal{T}_2$	B_4	6
$C_3 \times \mathcal{T}_s$	B_3	8

Remark 2. *We can also use the approach of [9, 10] as well as of the earlier work on the 2D Hénon-Heiles system [8] (see also [18]) to reconstruct qualitatively the nonlinear normal modes in theorem 1 using their isotropy groups. Figure 1 shows the projection of these periodic orbits in the configuration space* $\mathbf{R}^3$. *A detailed discussion is relegated to Appendix A.3, see also [14].*

In this work we will determine the actual number of nonlinear normal modes for each member of the family (3). Computing the value of H_ϵ (energy) for these modes and characterizing their linear stability we will give basic qualitative description of the whole parametric family (3).

1.2. Dynamical symmetry. Relative equilibria

An alternative proof of theorem 1 was suggested in [18] on the basis of the earlier work by Zhilinskií [19]. In order to follow this latter approach we would like to recall a number of known facts which we formulate as lemmas.

Lemma 1. *For all* $n > 0$ *the* n*-level set of the Hamiltonian* H_0 *in (3b) is a sphere* $\mathbf{S}^5_n : \{\xi \in T\mathbf{R}^3,\ H_0(\xi) = h_0 = n > 0\} \subset T\mathbf{R}^3 \backslash \{0\}$. *All orbits of the flow*

$\varphi_0 : (\mathbf{S}^5_n, t) \mapsto \mathbf{S}^5_n$ of the Hamiltonian vector field X_{H_0} are periodic with period 2π. This flow defines a symmetry group $\mathbf{S}^1$. The action of this symmetry on $T\mathbf{R}^3\backslash\{0\}$ and on $\mathbf{S}^5_n$ is free. The orbit space $\mathbf{S}^5_n/\mathbf{S}^1$ is a complex projective 2-space $\mathbf{CP}^2$.

Proof. Identify the phase space $T\mathbf{R}^3$ with a complex 3-space $\mathbf{C}^3$ with coordinates

$$w_1 = x + \mathrm{i}p_x, \quad w_2 = y + \mathrm{i}p_y, \quad \text{and } w_3 = z + \mathrm{i}p_z. \tag{4}$$

In these coordinates equation

$$H_0 = \tfrac{1}{2}(w_1\bar{w}_1 + w_2\bar{w}_2 + w_3\bar{w}_3) = h_0 = n > 0$$

defines a sphere $\mathbf{S}^5_n \subset \mathbf{C}^3\backslash\{0\}$ of radius $\sqrt{2n}$. The flow

$$\varphi_0 : (w, \bar{w}; t) \to (e^{it}w, e^{-it}\bar{w}) \tag{5}$$

is, obviously, diagonal, and all orbits are circles $\mathbf{S}^1_n$ of radius $\sqrt{2n}$. The quotient space $\mathbf{S}^5_n/\mathbf{S}^1_n$ is obtained by identifying points in each $\mathbf{S}^1_n \subset \mathbf{S}^5_n$ orbit. We come to one of the standard definitions of the complex projective space. See an appropriate textbook, for example [20]. $\square$

We denote by $\mathbf{CP}^2(n)$ the orbit space of H_0 for the level set $H_0^{-1}(n)$, $n > 0$. A convenient way to parameterize $\mathbf{CP}^2(n)$ globally is by using polynomial invariants of the flow φ_0.

Lemma 2. *Quadratic invariants of the $\mathbf{S}^1$ action (5) are*

$$\nu_1 = \tfrac{1}{2}w_1\bar{w}_1, \qquad \nu_2 = \tfrac{1}{2}w_2\bar{w}_2, \qquad \nu_3 = \tfrac{1}{2}w_3\bar{w}_3, \tag{6a}$$
$$\sigma_1 = \mathrm{Re}(w_2\bar{w}_3), \quad \sigma_2 = \mathrm{Re}(w_3\bar{w}_1), \quad \sigma_3 = \mathrm{Re}(w_1\bar{w}_2), \tag{6b}$$
$$\tau_1 = \mathrm{Im}(w_2\bar{w}_3), \quad \tau_2 = \mathrm{Im}(w_3\bar{w}_1), \quad \tau_3 = \mathrm{Im}(w_1\bar{w}_2), \tag{6c}$$

Except for the relation

$$\Sigma_0 = \nu_1 + \nu_2 + \nu_3 - n = 0, \tag{7a}$$

which fixes the level set $H_0^{-1}(n)$, these invariants are linearly independent. They satisfy nine algebraic relations $\Sigma_k = 0$, called syzygies *of the first order, where*

$$\Sigma_1 = 4\nu_2\nu_3 - \sigma_1^2 - \tau_1^2, \quad \Sigma_4 = 2\nu_1\sigma_1 - \sigma_2\sigma_3 + \tau_2\tau_3, \quad \Sigma_7 = 2\nu_1\tau_1 + \sigma_2\tau_3 + \tau_2\sigma_3, \tag{7b}$$
$$\Sigma_2 = 4\nu_3\nu_1 - \sigma_2^2 - \tau_2^2, \quad \Sigma_5 = 2\nu_2\sigma_2 - \sigma_3\sigma_1 + \tau_3\tau_1, \quad \Sigma_8 = 2\nu_2\tau_2 + \sigma_3\tau_1 + \tau_3\sigma_1, \tag{7c}$$
$$\Sigma_3 = 4\nu_1\nu_2 - \sigma_3^2 - \tau_3^2, \quad \Sigma_6 = 2\nu_3\sigma_3 - \sigma_1\sigma_2 + \tau_1\tau_2, \quad \Sigma_9 = 2\nu_3\tau_3 + \sigma_1\tau_2 + \tau_1\sigma_2. \tag{7d}$$

The syzygies (7) are themselves not algebraically independent.

The following two lemmas show that invariants (6) are used extensively in the reduction of the oscillator symmetry (5).

Lemma 3. *We can represent the points on $\mathbf{CP}^2(n)$, i.e., the orbits of the $\mathbf{S}^1$ action in (5) using $(\nu_1, \nu_2, \nu_3; \sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3)$ where the 9 parameters satisfy relations (7).*

Lemma 4. *Any $\mathbf{S}^1$ invariant smooth function $\mathbf{C}^3 \to \mathbf{R}$ is a smooth function of basic quadratic invariants (6). In particular, any $\mathbf{S}^1$ invariant polynomial can be expressed uniquely in terms of an integrity basis. One possible choice of such basis is*

$$\mathbb{R}[n, \nu_1 - \nu_2, \sigma_1, \sigma_2, \sigma_3] \bullet \{1, \nu_3, \nu_3^2, \tau_1, \tau_2, \tau_3\}.$$

Here $\mathbb{R}$ is the ring generated freely by the so-called principal invariants listed within the square brackets; auxiliary invariants listed within the curly brackets can enter only linearly, so that the whole ring can be represented as follows.

$$\mathbb{R}[n, \nu_1 - \nu_2, \sigma_1, \sigma_2, \sigma_3] \cdot 1 + \mathbb{R}[n, \nu_1 - \nu_2, \sigma_1, \sigma_2, \sigma_3] \cdot \nu_3 + \cdots$$

Proof. This lemma follows from a standard Gröbner basis computation. The structure of the polynomial ring is described by the following Molien generating function

$$g(\lambda) = (1 + 4\lambda + \lambda^2)/(1 - \lambda)^5, \tag{8}$$

where λ represents any of generators in (6), see more in [14]. □

Lemma 5. *Invariants (6) generate a Poisson algebra* su(3) *with n one of its Casimirs. The ring of invariant polynomials, generated multiplicatively by (6), can therefore be equipped with a Poisson structure. This structure is used to define Hamiltonian dynamical systems on* $\mathbf{CP}^2$.

Proof. The Poisson bracket $\{,\}$ of the two invariants in (6) is $\mathbf{S}^1$ invariant. By lemma 4 it can be expressed in terms of (6). Moreover, since (6) are all quadratic in (x, y, z, p_x, p_y, p_z), the brackets are linear in (6). The concrete Poisson structure is found straightforwardly by computing $\{,\}$ in the coordinates (x, y, z, p_x, p_y, p_z). The brackets satisfy relations of su(3). □

Lemma 6. *Near the limit of linearization* $\epsilon \to 0$ *the perturbed Hamiltonian* H_ϵ *in (3) is approximately invariant with respect to the flow* φ_0 *in lemma 1. Near* $\epsilon \to 0$ *we can* normalize H_ϵ *with respect to* H_0 *and make this approximate dynamical symmetry exact.*

Normalization deforms each trajectory of the system with Hamiltonian H_ϵ so that the flow of the normalized Hamiltonian $\widetilde{H}_\epsilon$ can be decomposed as $\varphi_0 \circ \widehat{\varphi} = \widehat{\varphi} \circ \varphi_0$; $\widetilde{H}_\epsilon$ and H_0 Poisson commute up to a certain order k in ϵ: $\{\widetilde{H}_\epsilon, H_0\} = \mathcal{O}(\epsilon^k)$.

Definition 1. *Relative equilibria (RE) are periodic orbits of the normalized system with Hamiltonian* $\widetilde{H}_\epsilon$ *in* $T\mathbf{R}^3$ *which are the group orbits of the* $\mathbf{S}^1$ *action in lemma 1.*

RE are also sometimes called *short periodic orbits*, i.e., periodic orbits with period 2π, or *basic orbits*.

To *reduce* the now exact $\mathbf{S}^1$ symmetry of $\widetilde{H}_\epsilon$, we pass from the original phase space $T\mathbf{R}^3$ to the space $\mathbf{CP}^2(n)$ of $\mathbf{S}^1_n$ orbits or the *reduced space* as follows. Since $\{\widetilde{H}_\epsilon, H_0\} = 0$, the value of $\widetilde{H}_\epsilon$ on each orbit of H_0 is constant. This means that we can properly define a function $\widehat{H}_\epsilon$ on the phase space $\mathbf{CP}^2(n)$ of H_0 by assigning to each $\mathbf{S}^1_n$ orbit of H_0 the value of $\widetilde{H}_\epsilon$ on any point of the orbit. We call the Hamiltonian $\widehat{H}_\epsilon$ on $\mathbf{CP}^2(n)$ the reduced Hamiltonian. Reduction results in the 2-DOF system on $\mathbf{CP}^2$ or the *reduced system*. By lemma 5, this system is a Poisson dynamical system with Hamiltonian $\widehat{H}$ expressed (uniquely) in terms of the invariants (6) and the integrity basis in lemma 4.

Lemma 7. *After reduction of the* $\mathbf{S}^1$ *symmetry, RE correspond to the equilibria of the reduced system with Hamiltonian* $\widehat{H}_\epsilon$ *on* $\mathbf{CP}^2$. *Linear Hamiltonian stability and isotropy group of the RE and of the corresponding stationary point of* $\widehat{H}_\epsilon$ *are the same.*

Proof. The first part of this lemma gives an alternative definition of relative equilibria. See, for example, Appendix 5C of [21] and Chap. 3.3 of [22]. □

The normal form of (3) is a formal power series whose orders are "tracked" by the degrees of the smallness parameter ϵ. Since this series diverges for typical values of parameters in (3), it is *truncated* at the order of intrest, which is in our case the principle order ϵ^2. As is well known (see for example Appendix 7 of [21]), the system described by such truncated normal form $\widetilde{H}_\epsilon$ and the original system are profoundly different. At the same time, it is possible to use $\widetilde{H}_\epsilon$ to analyze the short time average beaviour of the original system, and in particular its short periodic orbits. The adequateness (validity) of the truncated normal form approximation for the study of orbits of a given short period is clearly limited by the value of ϵ which should be sufficiently small. [Note that decreasing ϵ is equivalent to decreasing energy and approaching the equilibrium $x = y = z = 0$ of (3) where $H = 0$.] This makes $\widetilde{H}_\epsilon$ particularly suited for the analysis of the nonlinear normal modes which exist and can be studied anywhere close to the limit $\epsilon \to 0$.

Lemma 8. *As long as the normal form approximation is adequate, RE correspond to the nonlinear normal modes of the original system with Hamiltonian (3).*

Proof. **TO DO:** Uncover the relation of normalization and the Lyapunov–Schmidt reduction, see [23]. □

We conclude that the study of the nonlinear normal modes of the system with Hamiltonian (3) becomes the study of the equilibria of the reduced system, i.e., of the stationary points of the appropriately truncated reduced Hamiltonian $\widehat{H}_\epsilon$.

1.3. Symmetry and topology

In order to describe qualitatively the systems with Hamiltonian (3) in terms of their nonlinear normal modes, we should find the equilibria of Hamiltonians $\widehat{H}_\epsilon$ on $\mathbf{CP}^2$ (and hence the relative equilibria of the normalized system) and characterize them in terms of their energy and linear stability type. When searching for the stationary points of $\widehat{H}_\epsilon$ we should account for the action of $T_d \times \mathcal{T}$ on $\mathbf{CP}^2$ and the topology of this space.

Consider the action of a compact or finite group $\mathcal{G}$ on a manifold M. The isotropy group (or *stabilizer*) of $m \in M$ is the subgroup $\mathcal{G}_m$ of elements of $\mathcal{G}$ that leave m fixed. A point $m \in M$ is called a *fixed point* of the $\mathcal{G}$ action when $\mathcal{G}_m = \mathcal{G}$, that is, when it is fixed by all the elements of $\mathcal{G}$. The $\mathcal{G}$*-orbit* of m is the set $\mathcal{G}{\cdot}m = \{g \cdot m : g \in \mathcal{G}\}$. We are primarily interested in points $m_c \in M$ such that there is a neighbourhood of m_c in which there are no points m with isotropy group $\mathcal{G}_m$ which belongs to the same conjugacy class in $\mathcal{G}$ as $\mathcal{G}_{m_c}$. We call such points m_c and the orbit $\mathcal{G}{\cdot}m_c$ *critical*. For more details see, for example, [16]. The importance of the critical points is due to the following theorem by Louis Michel [24]:

Theorem. *Critical points of the action of a group $\mathcal{G}$ on a smooth manifold M are stationary points of every smooth, $\mathcal{G}$-invariant function H on M.*

Consequently, analysis of the critical points of the $T_d \times \mathcal{T}$ group action on the reduced space $\mathbf{CP}^2$ provides a number of relative equilibria of the normalized system and by lemma 8, nonlinear normal modes of the original system with Hamiltonian (3). A concrete study of this action results in the following conclusion (Zhilinskií [19], see also [18, 13, 14] and Appendix A).

Theorem 2. *The action of $T_d \times \mathcal{T}$ on $\mathbf{CP}^2$ induced by the action of $T_d \times \mathcal{T}$ on $T\mathbf{R}^3 \sim \mathbf{C}^3$ has 27 critical (i.e., isolated fixed) points grouped into five critical orbits with the conjugacy classes of stabilizers given in theorem 1. Table 1 presents position of these points on $\mathbf{CP}^2(n)$ characterized by the values of the invariants (6).*

Remark 3. *Table A3 of Appendix A lists isotropy groups of the 27 points in theorem 2. Isotropy groups of the points in the same critical orbit of the $T_d{\times}\mathcal{T}$ action are conjugate in $T_d \times \mathcal{T}$. These points are equivalent: dynamics in their neighborhood is identically* *the same, so it is usually sufficient to study one representative of each critical orbit. We denote different points of the same critical orbit by a superscript; we drop this index when referring to the entire orbit or when our results apply identically to all points in the orbit.*

point	$(\nu_1,\nu_2,\nu_3,\sigma_1,\sigma_2,\sigma_3,\tau_1,\tau_2,\tau_3)$
A_4^x	$n(1,0,0;0,0,0;0,0,0)$
A_4^y	$n(0,1,0;0,0,0;0,0,0)$
A_4^z	$n(0,0,1;0,0,0;0,0,0)$
A_3^a	$ns(\alpha,\alpha,\alpha;1,1,1;0,0,0)$
A_3^b	$ns(\alpha,\alpha,\alpha;\bar{1},\bar{1},1;0,0,0)$
A_3^c	$ns(\alpha,\alpha,\alpha;1,\bar{1},\bar{1};0,0,0)$
A_3^d	$ns(\alpha,\alpha,\alpha;\bar{1},1,\bar{1};0,0,0)$
A_2^x	$n(0,\alpha,\alpha;1,0,0;0,0,0)$
$A_2^{\bar{x}}$	$n(0,\alpha,\alpha;\bar{1},0,0;0,0,0)$
A_2^y	$n(\alpha,0,\alpha;0,1,0;0,0,0)$
$A_2^{\bar{y}}$	$n(\alpha,0,\alpha;0,\bar{1},0;0,0,0)$
A_2^z	$n(\alpha,\alpha,0;0,0,1;0,0,0)$
$A_2^{\bar{z}}$	$n(\alpha,\alpha,0;0,0,\bar{1};0,0,0)$

point	$(\nu_1,\nu_2,\nu_3,\sigma_1,\sigma_2,\sigma_3,\tau_1,\tau_2,\tau_3)$
B_4^x	$n(0,\alpha,\alpha;0,0,0;1,0,0)$
$B_4^{\bar{x}}$	$n(0,\alpha,\alpha;0,0,0;\bar{1},0,0)$
B_4^y	$n(\alpha,0,\alpha;0,0,0;0,1,0)$
$B_4^{\bar{y}}$	$n(\alpha,0,\alpha;0,0,0;0,\bar{1},0)$
B_4^z	$n(\alpha,\alpha,0;0,0,0;0,0,1)$
$B_4^{\bar{z}}$	$n(\alpha,\alpha,0;0,0,0;0,0,\bar{1})$
B_3^a	$ns(\alpha,\alpha,\alpha;\bar{\alpha},\bar{\alpha},\bar{\alpha};\beta,\beta,\beta)$
$B_3^{\bar{a}}$	$ns(\alpha,\alpha,\alpha;\bar{\alpha},\bar{\alpha},\bar{\alpha};\bar{\beta},\bar{\beta},\bar{\beta})$
B_3^b	$ns(\alpha,\alpha,\alpha;\alpha,\alpha,\bar{\alpha};\beta,\beta,\bar{\beta})$
$B_3^{\bar{b}}$	$ns(\alpha,\alpha,\alpha;\alpha,\alpha,\bar{\alpha};\bar{\beta},\bar{\beta},\beta)$
B_3^c	$ns(\alpha,\alpha,\alpha;\bar{\alpha},\alpha,\alpha;\bar{\beta},\beta,\beta)$
$B_3^{\bar{c}}$	$ns(\alpha,\alpha,\alpha;\bar{\alpha},\alpha,\alpha;\beta,\bar{\beta},\bar{\beta})$
B_3^d	$ns(\alpha,\alpha,\alpha;\alpha,\bar{\alpha},\alpha;\beta,\bar{\beta},\beta)$
$B_3^{\bar{d}}$	$ns(\alpha,\alpha,\alpha;\alpha,\bar{\alpha},\alpha;\bar{\beta},\beta,\bar{\beta})$

Table 1. Position of the fixed points of the $T_d \times \mathcal{T}$ action on $\mathbf{CP}^2(n)$ characterized by the values of the invariants (6). See table A3 for the isotropy groups of these points; $s=\frac{2}{3}$, $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}\sqrt{3}$, $\bar{\alpha}=-\alpha$, $\bar{\beta}=-\beta$, and $\bar{1}=-1$.

We can now prove theorem 1. We rely on lemma 6 in order to establish the correspondence of the nonlinear normal modes of the initial system with Hamiltonian (3) and the relative equilibria of the normalized system near the limit $\epsilon \to 0$. We then use the theorem of Michel and theorem 2.

Consider a smooth Hamiltonian function $\mathcal{H} : \mathbf{CP}^2 \to \mathbf{R}$ whose stationary points are nondegenerate. We call $\mathcal{H}$ *Morse type Hamiltonian.* The isotropy group of a stationary point c of $\mathcal{H}$ also restricts possible types of linear Hamiltonian stability and Morse index of c. Recall that linear stability is given by the eigenvalues of the 4×4 Hamiltonian matrix which describes the linearized equations of motion near $c \in \mathbf{CP}^2$, while the Morse index of c gives the number of negative eigenvalues of the Hessian matrix at c. Depending on the eigenvalues of this matrix we will distinguish six linear stability types EE, HH, EH, CH, 2E, 2H, described in Appendix B.2.

Theorem 3. *Equilibria of a $T_d \times \mathcal{T}$–invariant Morse type Hamltonian function on $\mathbf{CP}^2$ which are the critical points listed in theorem 2 can have the following linear Hamiltonian stability types and Morse indexes.*

critical orbit			stability	index
$D_{2d} \times \mathcal{T}$	A_4	3	2E	0, 4
			2H	2
$C_{3v} \times \mathcal{T}$	A_3	4	2E	0, 4
			2H	2
$C_{2v} \times \mathcal{T}$	A_2	6	EE	0, 2, 4
			EH	1, 3
			HH	2

critical orbit			stability	index
$S_4 \times \mathcal{T}_2$	B_4	6	EE	0, 2, 4
			EH	1, 3
$C_3 \times \mathcal{T}_s$	B_3	8	EE, CH	0, 2, 4

So far in this section we worked locally in an open neighbourhood $D_c \subset \mathbf{CP}^2$ of each critical point c of the $T_d \times \mathcal{T}$ action on $\mathbf{CP}^2$. As a result, our theorems cannot give any information on the total number of stationary points of $\mathcal{H}$. This information can only be obtained from the global (topological) analysis. Note, that we already used the topology of this space in order to find the action of $T_d \times \mathcal{T}$ (initially defined on $\mathbf{R}^3_{x,y,z}$) on $\mathbf{CP}^2$ and of G_c on D_c. In Appendix B we summarize how *Morse theory* [25, 26] is applied in order to check the consitency of any set of stationary points using four inequalities and one equality imposed by topology of $\mathbf{CP}^2$ on number and types of stationary points of Morse functions $\mathcal{H}$. In particular we can determine if it is possible for a $T_d \times \mathcal{T}$-invariant Morse function $\mathcal{H}$ to have stationary points solely at the critical points in theorem 2.

Lemma 9. *The simplest $T_d \times \mathcal{T}$-invariant Morse Hamiltonian $\mathcal{H}$ on $\mathbf{CP}^2$ has 27 equilibria which are critical points of the $T_d \times \mathcal{T}$ action in theorems 2 and 3. The six A_2 and six B_4 stationary points of $\mathcal{H}$ are of odd Morse index and have stability EH.*

Proof. According to theorem 3, points A_3, A_4, and B_3 are of even Morse index. The 27 points can have the right Morse indexes to give the Euler characteristics of $\mathbf{CP}^2$ *only* if A_2 and B_4 are of odd Morse index. □

Remark 4. *Following remark B.1, the simplest function $\mathcal{H}$ in lemma 9 can be also called the* perfect *$T_d \times \mathcal{T}$-invariant function on $\mathbf{CP}^2$.*

Remark 5. *Linear stability types of the nonlinear normal modes in theorem 1 correspond to the stability types in theorem 3. In the simplest possible case, the system with Hamiltonian (3) near the limit of linearization can have only these 27 families of periodic orbits.*

Remark 6. *Morse theory provides necessary conditions that must be satisfied by the stationary points of any Morse function on $\mathbf{CP}^2$. At the same time, when a set of known stationary points of a Morse function $\mathcal{H}$ obeys all these conditions, $\mathcal{H}$ can still have other stationary points.*

Remark 7. *A more complete consistency check of a known system of stationary points of a $T_d \times \mathcal{T}$ invariant Morse function $\mathcal{H}$ on $\mathbf{CP}^2$ requires satisfying Morse inequalities not only for $\mathbf{CP}^2$ but also for all $T_d \times \mathcal{T}$ invariant subspaces of $\mathbf{CP}^2$.*

1.4. Overview

We give a brief overview of this paper. In sec. 2, we continue a general study of $T_d \times \mathcal{T}$ symmetric systems with Hamiltonian (3) and show how these systems can be described as a one-parameter family. In section 3 we normalize the Hamiltonian (3) to the second (principal) order in ϵ (degree 4) and then reduce it. In sec. 4 we determine the local properties (linear stability type and Morse index) of the equilibria of the reduced Hamiltonian $\widehat{H}_\epsilon$ which are critical points of the $T_d \times \mathcal{T}$ action. In sec. 5 we describe other stationary points of $\widehat{H}_\epsilon$ which do not lie on a critical orbit of the $T_d \times \mathcal{T}$ action. This concludes the concrete study of the family of systems with Hamiltonian (3) near the limit $\epsilon \to 0$. Finally in sec. 6 we make some remarks about the bifurcations of the relative equilibria of this family. The paper contains two appendices where we detail the action of $T_d \times \mathcal{T}$ on $\mathbf{CP}^2$ and describe how we find the linear stability types and the Morse indexes of the critical points on $\mathbf{CP}^2$.

2. One-parameter classification

Theorem 1 has already highlighted the important consequences of the presence of the additional finite symmetry $T_d \times \mathcal{T}$. The next lemma shows that this symmetry causes important modifications of the standard $\mathbf{S}^1$–invariant polynomial basis in lemma 4.

Lemma 10. *Consider most general $(T_d \times \mathcal{T}) \times S^1$–invariant polynomials $P_k(w, \bar{w})$ of degree k in variables $(w, \bar{w})$. Then $P_k = 0$ if k is odd, and*

$$P_2 = c'n = c'(\nu_1 + \nu_2 + \nu_3) \tag{9a}$$

$$P_4 = cn^2 + a(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + b(\tau_1^2 + \tau_2^2 + \tau_3^2) \tag{9b}$$

where a, b, c, and c' are arbitrary constants.

Proof. Action of $\mathrm{O}(3) \times \mathcal{T}$ and its subgroup $T_d \times \mathcal{T}$ on polynomials (6) can be found by direct computation, see Appendix A and [14, 13]. In particular we can verify that all $\mathbf{S}^1$–invariants in (6) are invariant with respect to spatial inversion, and that

$$n, \quad (\tau_1, \tau_2, \tau_3), \quad \text{and} \quad \left(\sigma_1, \sigma_2, \sigma_3, \frac{3\nu_3 - n}{\sqrt{3}}, n_1 - n_2\right)$$

transform according to the irreducible representations of $\mathrm{O}(3)$ of indexes 0_g, 1_g and 2_g respectively. In other words, n and (τ_1, τ_2, τ_3) transform as a scalar and a axial 3-vector respectively. We can also easily verify that

$$\mathcal{T} : (\nu, \sigma, \tau) \to (\nu, \sigma, -\tau).$$

Knowing the action of $T_d \times \mathcal{T}$, we can further symmetrize the basis of lemma 4. In particular we obtain the generating function

$$g(\lambda) = \frac{1 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6 + \lambda^9}{(1-\lambda)(1-\lambda^2)^2(1-\lambda^3)(1-\lambda^4)} \tag{10}$$

which describes the symmetrized integrity basis, see [14]. The symmetrized invariants have high degrees in generators (6) : of the five denominator factors in (10), which describe principal invariants, $(1-\lambda)$ corresponds to n, while $(1-\lambda^2)^2$ represents two invariants of degree two in generators (6). Direct computation shows that τ^2 and σ^2 are $T_d \times \mathcal{T}$ invariant and can be chosen as these two invariants. The function (10) indicates that there are no other principal or auxiliary invariants of this degree. □

Remark 8. *The 3-vector $\tau = (\tau_1, \tau_2, \tau_3)$ is the angular momentum vector. Both n and $\tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2$ are totally symmetric with respect to the larger group $\mathrm{O}(3) \times \mathcal{T}$. The only term in (9) which represents $T_d \times \mathcal{T}$ symmetry is σ^2.*

We now come to a central result which provides the basis for our work.

Definition 2.

(i) Two reduced Hamiltonians $\widehat{H}_\epsilon$ and $\widehat{H}'_\epsilon$ will be called ϵ^2–equivalent if $\widehat{H}_\epsilon = a(n)\widehat{H}'_\epsilon + c(n) + \mathcal{O}(\epsilon^2)$, where $a(n)$ and $c(n)$ are functions of the parameter n.

(ii) Two normalized Hamiltonians $\widetilde{H}_\epsilon$ and $\widetilde{H}'_\epsilon$ are ϵ^2–equivalent if $\widehat{H}_\epsilon = a(H_0)\widehat{H}'_\epsilon + c(H_0) + \mathcal{O}(\epsilon^2)$, where a and c are functions of H_0 in (3b).

(iii) Two systems with Hamiltonians H_ϵ and H'_ϵ of the form (3) will also be called ϵ^2–equivalent if their respective normalized Hamiltonians are ϵ^2–equivalent.

Theorem 4. *At the level of ϵ^2–equivalence, all systems with Hamiltonian H_ϵ in (3) with generic values of K_3, K_0, and K_4, can be characterized using one single parameter.*

Proof. Lemma 6 establishes the correspondence of the initial system with Hamiltonian H_ϵ and the normalized system with Hamiltonian $\widetilde{H}_\epsilon$. According to lemma 10, $\widetilde{H}_\epsilon$ has the form $P_2 + \epsilon^2 P_4 + \ldots$. Since $n = H_0$ and therefore $\{n, \widetilde{H}_\epsilon\} = 0$, n is replaced by its value when we define the reduced Hamiltonian $\widehat{H}_\epsilon$. Then, up to the constant $c'n + \epsilon^2 cn^2$ and the overall scaling factor of ϵ^2,

$$\widehat{H} = K_s(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + K_t(\tau_1^2 + \tau_2^2 + \tau_3^2) + \ldots \quad . \tag{11}$$

It remains to verify that the family of systems (3) is generic in the sense that for practically all members of this family $K_s^2 + K_t^2 \neq 0$. (For the exceptional members we would have to normalize to higher orders.) This can be done in section 3 after computing explicitly the normal form $\widetilde{H}_\epsilon$ of (3). □

Remark 9. *Lemma 10 and theorem 4 apply, in fact, to a larger family of systems with an extended Hamiltonian $H_\epsilon + V'_\epsilon(x, y, z, p_x, p_y, p_z)$, where H_ϵ is defined in (3), and V'_ϵ is a general $T_d \times \mathcal{T}$–invariant ϵ-series perturbation of degree 3 or higher in all dynamical variables (x, y, z, p_x, p_y, p_z).*

Hamiltonian (11) is equivalent to the model quantum Hamiltonian proposed by Hecht [27] in order to describe the structure of polyads formed by triply degenerate vibrational modes of tetrahedral and octahedral molecules. Hecht's Hamiltonian was analyzed later by Patterson [28]. Both authors exploited the one-parameter property in their analysis. Classical RE system and quantum–classical comparison for this Hamiltonian is presented in sec. 11.1 of [14].

Remark 10. *In the generic situation when $R = \sqrt{K_s^2 + K_t^2} > 0$ we can define a one-parameter family by setting $K_s = R \sin\theta$ and $K_t = R\cos\theta$ and rescaling the reduced Hamiltonian $\widehat{H}$ in (11) by R. Then*

$$\widehat{H} = \sin\theta\,(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \cos\theta\,(\tau_1^2 + \tau_2^2 + \tau_3^2) + \ldots \ , \tag{12}$$

where in general, θ can take any value in $[0, 2\pi)$. Systems with the same value of θ but different values of R have qualitatively the same dynamics but different time scales. Specifically, for smaller R dynamics is slower.

To continue classifying reduced systems with Hamiltonian (12) and respective original systems with Hamiltonian (3) we use the equivalence relation which takes into account only the families of relative equilibria and respective nonlinear normal modes.

Definition 3. *Consider two different systems with Hamiltonians H_ϵ and H'_ϵ in (3).*

(i) Nonlinear normal modes of these two systems are equivalent if these modes have the same stability and the same isotropy group.

(ii) The sets of modes of the two systems are equivalent if we can establish a 1 : 1 correspondence between all modes in the sets using the equivalence above.

(iii) Two systems belong to the same class of mode–equivalent systems if their sets of nonlinear normal modes are equivalent.

Definition 4. *Consider two reduced systems with Hamiltonians $\widehat{H}_\epsilon$ and $\widehat{H}'_\epsilon$ in (12).*

(i) Equilibria of these two systems are equivalent if they have the same stability and the same isotropy group.

(ii) The sets of equilibria are equivalent if we can establish a 1 : 1 correspondence between the equivalent equilibria in the sets.

(iii) Two reduced systems are equivalent if their sets of equilibria are equivalent.

Consider now two reduced systems with Hamiltonians $\widehat{H}_\epsilon$ and $\widehat{H}'_\epsilon$ which correspond to the systems in definition 3. By lemma 8 we obtain

Lemma. *Two systems belong to the same class of mode–equivalent systems in definition 3 if the corresponding reduced systems are equivalent in the sense of definition 4.*

Proof. Relate the equilibria of the reduced system, the relative equilibria of the normalized system and the nonlinear normal modes of the original system. □

Definition 5. *The system with Hamiltonian H_ϵ in (3) is called ϵ^2–generic if it is mode–equivalent to the respective normalized system with Hamiltonian $\widetilde{H}_\epsilon$ truncated at the principal order ϵ^2.*

In this work we will only consider ϵ^2–generic systems with Hamiltonian (3). In order to characterize all such systems we can study all possible sets of stationary points $\xi \in \mathbf{CP}^2(n)$ of the Hamiltonian (12). Symmetry properties of ξ can be obtained from the study of the action of $T_d \times \mathcal{T}$ on $\mathbf{CP}^2(n)$. Stability of ξ is given by four eigenvalues $(\pm\lambda_1, \pm\lambda_2)$ of the Hamiltonian matrix of the locally linearized Hamiltonian $\widehat{H}\big|_\xi$. We also use the eigenvalues of the Hessian matrix to compute the Morse index of ξ.

Remark 11. *It is sufficient to study (12) for $\theta \in [0, \pi)$ because $\widehat{H}(\theta+\pi) = -\widehat{H}(\theta)$. Both Hamiltonian and Hessian matrices change signs when $\theta \to \theta + \pi$. This does not affect stability (since both λ and $-\lambda$ are eigenvalues). On the other hand, if the Morse index for θ is d then for $\theta + \pi$ it becomes $4 - d$.*

We like to point out that our classification has both qualitative and quantitative aspects. We will find several different classes of ϵ^2 generic systems with Hamiltonian (3). At the same time we represent all systems in the same class as a continuous one-parameter family and describe quantitatively the evolution of their relative equilibria.

Remark 12. *It is instructive to compare our analysis to a similar single parameter classification of the 2D Hénon-Heiles systems with Hamiltonian (2) which is summarized in Appendix C. The principal difference between our case and the 2D case is that systems with different θ in (12) have the same principal action n and do not differ seriously in energy. Figuratively, our scheme is "horizontal" or "isoenergetic", while the only possible similar scheme for the 2D systems is "vertical".*

3. Normalization and reduction

Normalization is the first step of the concrete study of systems with Hamiltonian H_ϵ in (3). This well known procedure can be performed using the Lie series method [29, 30, 31]. To the second order terms ϵ^2 we obtain the normalized Hamiltonian

$$\widetilde{H}_\epsilon(w,\bar{w}) = \widetilde{H}_0(w,\bar{w}) + \epsilon^2 \widetilde{H}_2(w,\bar{w}) \;, \tag{13a}$$

where variables $(w,\bar{w})$ are given in (4), $\widetilde{H}_0(w,\bar{w}) = H_0(w,\bar{w}) = \frac{1}{2}\sum w_i\bar{w}_i$, and

$$\begin{aligned}\widetilde{H}_2(w,\bar{w}) &= -\tfrac{3}{2}(K_0+K_4)(w_1^2\bar{w}_1^2 + w_2^2\bar{w}_2^2 + w_3^2\bar{w}_3^2)\\ &+ (-2K_0 + \tfrac{1}{6}K_3^2)(w_1\bar{w}_1w_2\bar{w}_2 + w_2\bar{w}_2w_3\bar{w}_3 + w_3\bar{w}_3w_1\bar{w}_1)\\ &+ (-\tfrac{1}{2}K_0 + \tfrac{1}{8}K_3^2)(w_1^2\bar{w}_2^2 + \bar{w}_1^2w_2^2 + w_2^2\bar{w}_3^2 + \bar{w}_2^2w_3^2 + w_3^2\bar{w}_1^2 + \bar{w}_3^2w_1^2) \;.\end{aligned} \tag{13b}$$

Reduction of the Hamiltonian (13) gives

$$\widehat{H}_\epsilon = \widehat{H}_0 + \epsilon^2\widehat{H}_2 \tag{14a}$$

where

$$\widehat{H}_0 = \nu_1 + \nu_2 + \nu_3 = n \tag{14b}$$

$$\widehat{H}_2 = \tfrac{3}{2}(K_4 + K_0)n^2 + K_s(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + K_t(\tau_1^2 + \tau_2^2 + \tau_3^2) \tag{14c}$$

with

$$K_s = (-5K_3^2 - 36K_4)/48, \quad K_t = (K_3^2 - 36K_4 - 24K_0)/48 \,. \tag{14d}$$

Ignoring the constant terms in $\widehat{H}_\epsilon$ and rescaling by ϵ^2 we arrive at the Hamiltonian $\widehat{H}$ in (11). Furthermore, since K_0, K_3 and K_4 can take arbitrary values (of order 1) we can rewrite $\widehat{H}$ in the one-parameter form $\widehat{H}(\theta)$ in (12), where $0 \le \theta < \pi$.

Remark 13. *There are two values of θ at which the reduced system with Hamiltonian $\widehat{H}(\theta)$ has large Lie group of symmetry and is Liouville integrable, see below.*

first integrals	symmetry	value of θ
$n,\ \tau_1^2 + \tau_2^2 + \tau_3^2,\ \tau_3$	O(3)	0
$\nu_1,\ \nu_2,\ \nu_3$	SU(3)	$\pi/4$

Remark 14. *Including the terms μ_4 and μ_2^2 in (3) results in the most general $T_d \times \mathcal{T}$ invariant fourth order reduced system. With $K_4 = K_0 = 0$ and $K_3 \neq 0$ in (3) the reduced Hamiltonian is*

$$\widehat{H} = -5K_3^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + K_3^2(\tau_1^2 + \tau_2^2 + \tau_3^2),$$

which represents only one member of the family (12) with $\theta = \pi - \tan^{-1}(5) \approx 0.563\pi$.

4. Relative equilibria corresponding to critical points

We study equilibria of the reduced system with Hamiltonian (12) which are the critical points of the $T_d \times \mathcal{T}$ action on $\mathbf{CP}^2(n)$.

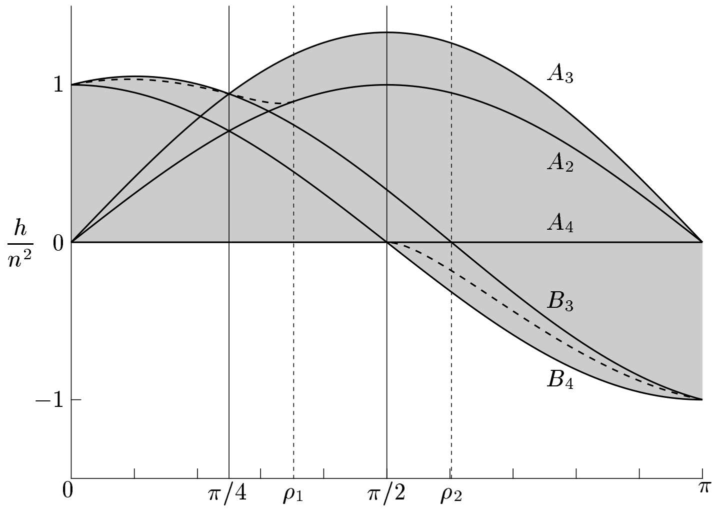

Figure 2. Scaled energy h/n^2 of the stationary points of $\widehat{H}$ (12) as a function of the parameter θ. The dashed curve marks the energy of the C_s point (see sec. 5).

Lemma 11. *The nonlinear normal modes in theorem 1 correspond to the relative equilibria (RE) of the normalized system with Hamiltonian (3). On the phase space* $\mathbf{CP}^2$ *of the corresponding reduced system, these RE are critical points of the action of the symmetry group* $T_d \times \mathcal{T}$ *given in theorem 2. The principal terms in the energy–action characteristics for these modes are given below.*

conjugacy class of stabilizers	shorthand notation	number of modes	energy $\widehat{H}(n,\theta)$
$D_{2d} \times \mathcal{T}$	A_4	3	0
$C_{3v} \times \mathcal{T}$	A_3	4	$\frac{4}{3}n^2 \sin\theta$
$C_{2v} \times \mathcal{T}$	A_2	6	$n^2 \sin\theta$
$S_4 \times \mathcal{T}_2$	B_4	6	$n^2 \cos\theta$
$C_3 \times \mathcal{T}_s$	B_3	8	$\frac{1}{3}n^2(\sin\theta + 3\cos\theta)$

Here n *is the action* $I = \oint p dq$ *computed along the respective periodic orbit, and* θ *is defined in remark 10. For the members of the family of systems with Hamiltonian (3), the absolute maximum and minimum accessible energy* $E_{\max}(n)$ *and* $E_{\min}(n)$ *for a given fixed action* n *can be estimated as follows:* $E_{\min} = E_{A_4}$ *and* $E_{\min} = E_{B_4}$ *in the regions* $\theta \in [0, \frac{1}{2}\pi]$ *and* $\theta \in [\frac{1}{2}\pi, \pi]$ *respectively, while* $E_{\max} = E_{B_3}$ *and* $E_{\max} = E_{A_3}$ *in the regions* $\theta \in [0, \frac{1}{4}\pi]$ *and* $\theta \in [\frac{1}{4}\pi, \pi]$ *respectively.*

Proof. We find the energy for each type of RE by substituting the coordinates in table 1 into (12). Figure 2 and table 2 present the resulting full view of the one-parameter family (12). We now prove that $\widehat{H}/n^2$ takes the values represented by the gray region in fig. 2. We do this in a number of steps. First, note that when $0 \le \theta \le \pi/2$ we have $\sin\theta \ge 0$ and $\cos\theta \ge 0$, therefore $\widehat{H} \ge 0 = \widehat{H}_{A_4}$ in that region.

Let $\nu^2 \equiv \nu_1^2 + \nu_2^2 + \nu_3^2$, $\sigma^2 \equiv \sigma_1^2 + \sigma_2^2 + \sigma_3^2$ and $\tau^2 \equiv \tau_1^2 + \tau_2^2 + \tau_3^2$. From (7a) we find that $\nu^2 \ge n^2/3$. In the region $\pi/4 \le \theta \le \pi$ we express $\widehat{H}$ as $\widehat{H} = \sin\theta(\sigma^2+\tau^2)+(\cos\theta-\sin\theta)\tau^2$. Using the syzygies (7) we find that $\sigma^2+\tau^2 = 2(n^2-\nu^2)$ and since $\nu^2 \ge n^2/3$ we get $\sigma^2 + \tau^2 \le 4n^2/3$. Also when $\pi/4 \le \theta \le \pi$ we have $\cos\theta - \sin\theta \le 0$. Therefore $\widehat{H} \le \sin\theta(\sigma^2 + \tau^2) \le 4n^2 \sin\theta/3 = \widehat{H}_{A_3}$.

In order to complete the argument we need to show that $\tau^2 \le n^2$. By remark 8, any rotation in the original phase space $T\mathbf{R}^3$ leaves τ^2 unchanged. Note also that the form of the syzygies in (7) remains invariant under such rotation. Therefore we can rotate coordinate axes so that in the new coordinate system we have $\tau_1' = \tau_2' = 0$ and ${\tau_3'}^2 = \tau^2$. If $\tau_3' = 0$, then what we want to prove is true. If $\tau_3' \neq 0$ then using the syzygies we find that $\nu_3' = \sigma_1' = \sigma_2' = 0$ and ${\tau_3'}^2 = 4\nu_1'(n - \nu_1') - {\sigma_3'}^2$. It follows from the last relation that ${\tau_3'}^2$ and therefore τ^2 is less or equal than n^2.

We complete the proof. In the region $\pi/2 \le \theta \le \pi$ we have that $\widehat{H} = \sin\theta\sigma^2 - |\cos\theta|\tau^2$. Since $\sin\theta \ge 0$ we get $\widehat{H} \ge -|\cos\theta|\tau^2 \ge -|\cos\theta|n^2 = \widehat{H}_{B_4}$. Finally, in the region $0 \le \theta \le \pi/4$ we have $\widehat{H} = \sin\theta(\sigma^2 + \tau^2) + (\cos\theta - \sin\theta)\tau^2 \le \frac{4n^2}{3}\sin\theta + n^2(\cos\theta - \sin\theta) = \frac{n^2}{3}(\sin\theta + 3\cos\theta) = \widehat{H}_{B_3}$. □

We now study linear stability of the RE found in lemma 11 and the Morse index of the coresponding stationary points. Theorem 3 leaves a number of different possibilities which require a concrete study of the Hamiltonain (12).

Lemma 12. *The one-parameter family of reduced Hamiltonians (12) and corresponding Hamiltonians (3) can be separated into five qualitatively different subfamilies, which corespond to the five open intervals of the values of parameter* θ*. Concrete values are listed in table 2. Each subfamily is distinguished by a particular pattern of the linear stability of the relative equilibria in lemma 11.*

region		$E_{\min}$	$E_{\max}$	A_4	A_3	A_2	B_4	B_3	C_s
I	$(0, \frac{1}{4}\pi)$	$\widehat{H}_{A_4}$	$\widehat{H}_{B_3}$	2E 0	2H 2	EH 1	EE 2	EE 4	EH 3
IIa	$(\frac{1}{4}\pi, \rho_1)$	$\widehat{H}_{A_4}$	$\widehat{H}_{A_3}$	2E 0	2E 4	EE 2	EH 1	CH 2	EH 3
IIb	$(\rho_1, \frac{1}{2}\pi)$			2E 0	2E 4	EH 3	EH 1	CH 2	
IIIa	$(\frac{1}{2}\pi, \rho_2)$	$\widehat{H}_{B_4}$	$\widehat{H}_{A_3}$	2H 2	2E 4	EH 3	EE 0	CH 2	EH 1
IIIb	(ρ_2, π)			2H 2	2E 4	EH 3	EE 0	EE 2	EH 1

Table 2. Stability (2E, 2H, etc, as explained in Appendix B.2) and Morse index $(0 \ldots 4)$ of the stationary points of $\widehat{H}$ in (12). For the C_s points see sec. 5; $\rho_1 = \cos^{-1}(1/\sqrt{5})$ and $\rho_2 = \cos^{-1}(-1/\sqrt{10})$.

A_4^x	$\chi_1 = -\frac{1}{\sqrt{2n}}\sigma_2$	$\chi_2 = \frac{1}{\sqrt{2n}}\sigma_3$
	$\psi_1 = \frac{1}{\sqrt{2n}}\tau_2$	$\psi_2 = \frac{1}{\sqrt{2n}}\tau_3$
	$\nu_1 = \delta/2$	$\nu_2 = (\sigma_3^2 + \tau_3^2)/2\delta$
	$\nu_3 = (\sigma_2^2 + \tau_2^2)/2\delta$	$\sigma_1 = (\sigma_2\sigma_3 - \tau_2\tau_3)/\delta$
	$\tau_1 = (-\sigma_2\tau_3 - \tau_2\sigma_3)/\delta$	
	where $\delta = n + (n^2 - \sigma_2^2 - \sigma_3^2 - \tau_2^2 - \tau_3^2)^{1/2}$	
A_3^a	$\chi_1 = \frac{1}{2^{1/4}\sqrt{n}}(2n - 2\sigma_2 - \sigma_3)$	$\chi_2 = -\frac{1}{2^{1/4}\sqrt{3n}}(2n - 3\sigma_3)$
	$\psi_1 = \frac{3}{2^{7/4}\sqrt{n}}\tau_3$	$\psi_2 = \frac{\sqrt{3}}{2^{7/4}\sqrt{n}}(2\tau_2 + \tau_3)$
	$\nu_1 = \delta/2$	$\nu_2 = (\sigma_3^2 + \tau_3^2)/2\delta$
	$\nu_3 = (\sigma_2^2 + \tau_2^2)/2\delta$	$\sigma_1 = (\sigma_2\sigma_3 - \tau_2\tau_3)/\delta$
	$\tau_1 = (-\sigma_2\tau_3 - \tau_2\sigma_3)/\delta$	
	where $\delta = n - (n^2 - \sigma_2^2 - \sigma_3^2 - \tau_2^2 - \tau_3^2)^{1/2}$	
A_2^z	$\chi_1 = \frac{1}{\sqrt{n}}(\nu_2 - \frac{n}{2})$	$\chi_2 = \frac{1}{\sqrt{n}}\sigma_1$
	$\psi_1 = \frac{1}{\sqrt{n}}\tau_3$	$\psi_2 = \frac{1}{\sqrt{n}}\tau_1$
	$\nu_1 = (\tau_3^2 + \delta^2)/4\nu_2$	$\nu_3 = (\sigma_1^2 + \tau_1^2)/4\nu_2$
	$\sigma_2 = (-\tau_1\tau_3 + \sigma_1\delta)/2\nu_2$	$\sigma_3 = \delta$
	$\tau_2 = (-\sigma_1\tau_3 - \tau_1\delta)/2\nu_2$	
	where $\delta = (4n\nu_2 - 4\nu_2^2 - \sigma_1^2 - \tau_1^2 - \tau_3^2)^{1/2}$	
B_4^x	$\chi_1 = \frac{1}{\sqrt{n}}\sigma_1$	$\chi_2 = \frac{1}{\sqrt{n}}\sigma_2$
	$\psi_1 = \frac{1}{\sqrt{n}}(\nu_3 - \frac{n}{2})$	$\psi_2 = \frac{1}{\sqrt{n}}\tau_2$
	$\nu_1 = (\sigma_2^2 + \tau_2^2)/4\nu_3$	$\nu_2 = (\sigma_1^2 + \delta)/4\nu_3$
	$\sigma_3 = (\sigma_1\sigma_2 - \tau_2\delta)/2\nu_3$	$\tau_1 = \delta$
	$\tau_3 = (-\sigma_1\tau_2 - \sigma_2\delta)/2\nu_3$	
	where $\delta = (4n\nu_3 - 4\nu_3^2 - \sigma_1^2 - \sigma_2^2 - \tau_2^2)^{1/2}$	
B_3^a	$\chi_1 = \frac{1}{4\sqrt{n}}(\sqrt{3}(\sigma_2 - \sigma_3) + \tau_2 - \tau_3)$	$\chi_2 = \frac{1}{4\sqrt{n}}(3(\sigma_2 + \sigma_3) + \sqrt{3}(\tau_2 + \tau_3))$
	$\psi_1 = -\frac{1}{2\sqrt{n}}(4n + 3\sigma_2 - \sqrt{3}\tau_2 - 2\sqrt{3}\tau_3)$	$\psi_2 = -\frac{1}{2\sqrt{n}}(\sqrt{3}\sigma_2 + 2\sqrt{3}\sigma_3 + 3\tau_2)$
	$\nu_1 = \delta/2$	$\nu_2 = (\sigma_3^2 + \tau_3^2)/2\delta$
	$\nu_3 = (\sigma_2^2 + \tau_2^2)/2\delta$	$\sigma_1 = (\sigma_2\sigma_3 - \tau_2\tau_3)/\delta$
	$\tau_1 = (-\sigma_2\tau_3 - \tau_2\sigma_3)/\delta$	
	where $\delta = n - (n^2 - \sigma_2^2 - \sigma_3^2 - \tau_2^2 - \tau_3^2)^{1/2}$	

Table 3. Local coordinates at critical points on $\mathbf{CP}^2(n)$. We use these relations in order to express the Hamiltonian in each local chart and to compute the Poisson brackets of the local coordinates.

Proof. According to remark 3, it suffices to study one critical point for each of the five critical orbits of the $T_d \times \mathcal{T}$ action on $\mathbf{CP}^2(n)$. We begin by finding an appropriate local symplectic chart in the neighborhood of the critical point, and then compute the quadratic part of the Hamiltonian (12) in this local chart. We define the charts $(\chi_1, \chi_2, \psi_1, \psi_2)$ in terms of the polynomial invariants as described in detail in Appendix B. These charts are given in table 3. Note that our local coordinates (χ, ψ) are canonical only up to the constant terms in the Poisson brackets,

$$\{\chi_k, \psi_k\} = 1 + \dots, \quad \{\chi_1, \psi_2\} = 0 + \dots, \quad \{\chi_2, \psi_1\} = 0 + \dots, \quad k = 1, 2.$$

This is adequate only for the study of the linearized equations of motion. We now express Hamiltonian (12) in each local chart and expand it to the second order terms. This gives

$$\begin{aligned}
&H_{A^4}(\chi, \psi) = 2n \cos\theta(\psi_1^2 + \psi_2^2) + 2n \sin\theta(\chi_1^2 + \chi_2^2),\\
&H_{A^3}(\chi, \psi) = \tfrac{4}{3} n^2 \sin\theta + \tfrac{4\sqrt{2}}{3} n(\cos\theta - \sin\theta)(\psi_1^2 + \psi_2^2) - \sqrt{2} n \sin\theta(\chi_1^2 + \chi_2^2);\\
&H_{A^2}(\chi, \psi) = n^2 \sin\theta + n((\cos\theta - \sin\theta)\psi_1^2 + (2\cos\theta - \sin\theta)\psi_2^2 - 4\sin\theta\chi_1^2 + \sin\theta\chi_2^2);\\
&H_{B^4}(\chi, \psi) = n^2 \cos\theta + n(-4\cos\theta\psi_1^2 + \sin\theta\psi_2^2 - (\cos\theta - \sin\theta)\chi_1^2 + \sin\theta\chi_2^2);\\
&H_{B^3}(\chi, \psi) = \tfrac{1}{3} n\big[(\sin\theta + 3\cos\theta)\big(n - (\psi_1^2 + \psi_2^2) - 2\sqrt{3}(\chi_1\psi_1 + \chi_2\psi_2)\big)\\
&\qquad\qquad - 12\cos\theta(\chi_1^2 + \chi_2^2) + 6(\sin\theta - \cos\theta)(\chi_1\psi_2 - \chi_2\psi_1)\big].
\end{aligned}$$

(Note that in full agreement with remark 3, we can always find such two local charts for any two critical points in the same critical orbit, that the respective local Hamiltonians are the same.) Using local quadratic Hamiltonians $H(\chi, \psi)$ we compute the Morse indexes and the linear stability types in table 2. □

Remark 15. *Besides the linear stability type of the stationary points of $\widehat{H}$ we can also determine that in some cases the stationary point is* orbitally stable. *This is true when the Morse index of the point is either* 0 *or* 4 *and we can use the reduced Hamiltonian to get the appropriate estimate.*

Corollary 1. 27 *equilibria of the system with Hamiltonian $\widehat{H}$ in (12) which are the critical points of the $T_d \times \mathcal{T}$ action on $\mathbf{CP}^2$ satisfy Morse inequalities and give the right Euler characteristics for $\mathbf{CP}^2$ only in the region IIb when $\theta \in (\rho_1, \frac{1}{2}\pi)$. In this region $\widehat{H}$ can be the simplest $T_d \times \mathcal{T}$ invariant Morse function. For all other values of θ this system must have other equilibria.*

Proof. Use lemma 9, Appendix B, and table 2. □

Proposition 1. *For θ in the interval $(\rho_1, \frac{1}{2}\pi)$ the function $\widehat{H}$ in (12) is the simplest (or perfect) $T_d \times \mathcal{T}$ invariant Morse function in lemma 9 whose stationary points are described in lemma 12.*

We do not prove the above proposition in this paper. However, all our subsequent analysis is in full agreement with it.

5. Relative equilibria corresponding to non-critical points

In this section, we find additional equilibria of the system with Hamiltonian $\widehat{H}$ in (12) predicted in corollary 1 for θ outside the closed interval $[\rho_1, \frac{1}{2}\pi]$. These additional equilibria ξ are likely to be created in bifurcations which take place when the value of θ leaves the interval $(\rho_1, \frac{1}{2}\pi)$. This is even more likely if proposition 1 is true.

Remark 16. *Let c be one of the critical points of the $T_d \times \mathcal{T}$ action on $\mathbf{CP}^2$ described in theorem 2. By the theorem of Michel cited in sec. 1.3, c is a stationary point of $\widehat{H}(\theta)$ in (12). Suppose that the new stationary point ξ is created in a bifurcation of c. When considering where ξ can be found, we should take into account the following:*

(i) *The isotropy group G_ξ of ξ is a subgroup of the isotropy group G_c of c.*

(ii) *If c belongs not only in the closure of the stratum with trivial stabilizer C_1 but also in the closure of one or several strata with nontrivial stabilizers G', G'', etc, than a generic one-parameter bifurcation of the stationary point c will not break the symmetry G_c all the way down to C_1 and G_ξ will become one of G', G'', etc.*

(iii) *If such a generic bifurcation takes place, the new point ξ with non-trivial stabilizer G_ξ will remain on a continuous set $\mathfrak{M} \subset \mathbf{CP}^2$ of non-critical G_ξ–invariant points. $\mathfrak{M}$ can in turn be a subset of a submanifold $M \subset \mathbf{CP}^2$ with isotropy group $G_M \subseteq G_\xi$.*

(iv) *The manifold $M \ni \xi$ can contain points c of higher isotropy; it can itself be a subspace of a larger manifold $M' \subset \mathbf{CP}^2$ with lower isotropy group $G_{M'} \subset G_M \subseteq G_\xi \subset G_c$.*

(v) *When looking for stationary points $\xi \in M \subset M' \subset \ldots \mathbf{CP}^2$ we should check Morse theory requirements for all invariant submanifods M, M', etc.*

(vi) *Particularly interesting are situations when $P = M$, or M', etc is also dynamically invariant because G_P is symplectic. In that case P is symplectic, we can restrict our system on P and look for its equilibria there.*

(vii) *The action of $T_d \times \mathcal{T}$ on $\mathbf{CP}^2$ has several invariant submanifolds M with topology $\mathbf{S}^1$, $\mathbf{S}^2 \sim \mathbf{CP}^1$, and $\mathbf{RP}^2$. Information on these manifolds and their intersections can be found in [14] and Appendix A. The 2-spheres with stabilizers of conjugacy class C_s and C_2 can serve as restricted phase spaces P.*

5.1. Existence and stability of the $C_s \wedge \mathcal{T}_2$ relative equilibria

Following remark 16(vi–vii), consider the C_s and C_2–invariant spheres $\mathbf{S}^2 \subset \mathbf{CP}^2(n)$ described in Appendix A. Critical points of type A_4 and A_2 are intersection points of the two types of spheres (see fig. A2). This means that the new equilibrium points ξ created in a bifurcation of A_4 or A_2 can depart either on a C_2 or a C_s sphere, cf. remark 16(iii). On the C_s spheres we also find points of type B_4, while on the C_2 spheres we find points of type A_3.

Lemma 13. *The A_2, A_4, and B_4 equilibrium points of the system with Hamiltonian $\widehat{H}$ in (12) do not alone give the right Euler characteristics for $\mathbf{S}^2$ on the C_s spheres when $0 < \theta < \rho_1$ and $\frac{1}{2}\pi < \theta < \pi$, i.e., outside the region IIb in table 2; they give the right characteristics when $\rho_1 < \theta < \frac{1}{2}\pi$. Furthermore, for the same values of θ, the two points A_2 and A_4, which lie on the same $C_s \wedge \mathcal{T}_2$ –invariant circle of the C_s sphere, do not alone give the correct Euler characteristics for $\mathbf{S}^1$.*

Proof. Local coordinates in table 3 are chosen so, that they always split into C_s and C_2 invariant symplectic pairs. This makes checking Morse inequalities and Euler characteristics for the C_2 and C_s spheres straightforward. Alternatively, we can restrict the vector field $X_{\widehat{H}}$ on the spheres and look for stationary points. (Since the spheres are dynamically invariant, any equilibrium of the restricted system is also an equilibrium on the full system.) □

Corollary 2. *In all regions in table 2 except IIb, the system with Hamiltonian (12) should have additional equilibria $\xi \in \mathbf{S}^1 \subset \mathbf{S}^2 \subset \mathbf{CP}^2(n)$, where the isotropy group of ξ and $\mathbf{S}^1$ is $C_s \wedge \mathcal{T}_2$ and $\mathbf{S}^2$ has isotropy group C_s.*

Lemma 14. *The $C_s \wedge \mathcal{T}_2$–invariant equilibria in corollary 2 exist only when*

$$0 < \theta < \rho_1 = \cos^{-1}(1/\sqrt{5}), \quad or \quad \tfrac{1}{2}\pi < \theta < \pi,$$

and have stability type EH. By symmetry, there should be two such equilibria on each of the six C_s spheres. The original system with Hamiltonian (3) has 12 corresponding $C_s \wedge \mathcal{T}_2$–invariant nonlinear normal modes.

Proof. By the argument similar to that in remark 3, it is sufficient to study one of the six equivalent C_s spheres. We chose the sphere C_s^{ab} which is defined in Appendix A as the set of points on $\mathbf{CP}^2(n)$ whose coordinates $(\nu_1, \nu_2, \nu_3; \sigma_1, \sigma_2, \sigma_3; \tau_1, \tau_2, \tau_3)$ are

$$(\nu; \sigma; \tau) = n\left(\frac{1+w}{4}, \frac{1+w}{4}, \frac{1-w}{2}; \frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}}, \frac{1+w}{2}; \frac{v}{\sqrt{2}}, -\frac{v}{\sqrt{2}}, 0\right), \tag{15}$$

where $u^2 + v^2 + w^2 = 1$. The $C_s \wedge \mathcal{T}_2$ invariant circle on this sphere is defined by the additional equation $u = 0$. The reduced system restricted to this sphere corresponds to the original system restricted to the 4-plane in $T\mathbf{R}^3$ defined by $\{x = y,\ p_x = p_y\}$. The Poisson algebra generated by the functions (u, v, w) in (15) and restricted to the

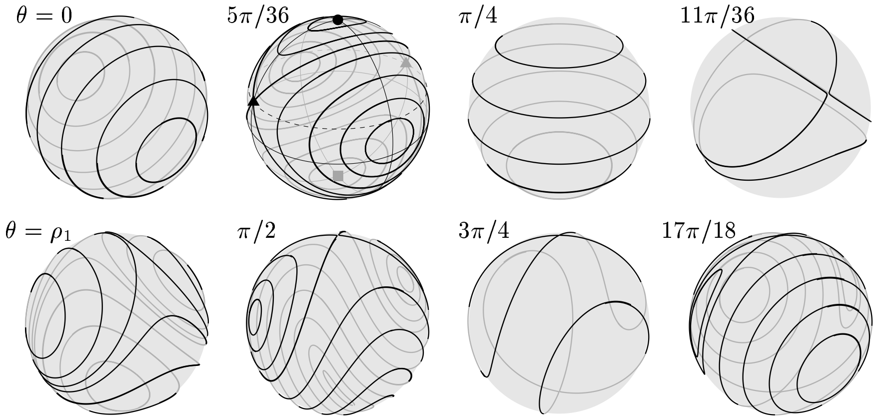

Figure 3. Phase portraits of the system with Hamiltonian $\widehat{H}$ in (12) restricted to the C_s^{ab} invariant sphere $\mathbf{S}^2 \subset \mathbf{CP}^2(n)$ for different values of the parameter θ. The sphere is oriented in the same way as in fig. A3b; location of the critical points A_2, A_4, and A_3 is marked for the $\theta = 5\pi/36$ case by a filled disc, triangle, and square respectively.

C_s^{ab} sphere is the algebra so(3) with Casimir $u^2 + v^2 + w^2$. We find the vector field of Hamiltonian (12) restricted to this sphere

$$\begin{aligned} \dot{u} &= -v(w+1)(\sin\theta - 4\cos\theta) - 4v\cos\theta \\ \dot{v} &= u(1-3w)\sin\theta \\ \dot{w} &= 4uv(\sin\theta - \cos\theta) \end{aligned} \tag{16}$$

The constant level sets of this system are shown in fig. 3 in the coordinate system of fig. A3b with axis w aligned vertically. The equations $\dot{u} = \dot{v} = \dot{w} = 0$ for the equilibria on the sphere can be easily solved. Solutions $u = 0, v = 0, w = 1$ (North pole) and $u = 0, v = 0, w = -1$ (South pole) represent points $A_2^{\tilde{z}}$ and $A_4^{\tilde{z}}$ respectively; the two points A_3^a and A_3^b correspond to $u = \pm\frac{2}{3}\sqrt{2}, v = 0, w = \frac{1}{3}$. Positions of these critical points is shown in fig. 3 for the example of $\theta = 5\pi/36$. The two new stationary points $\xi_\pm$ have coordinates

$$u = 0, \quad v = \pm(1-w^2)^{1/2}, \quad w = \frac{\sin\theta}{4\cos\theta - \sin\theta}.$$

This solution exists only for the values of θ specified in the lemma. From (15) we find the $\mathbf{CP}^2(n)$ coordinates of $\xi_\pm$

$$\xi_\pm = n\big(r, r, 1-2r; 0, 0, 2r; \pm 2\sqrt{r(1-2r)}, \pm 2\sqrt{r(1-2r)}, 0\big)$$

where $r = \cos\theta/(4\cos\theta - \sin\theta)$. In fig. 3, $\theta = 5\pi/36$ (region I), $\xi_\pm$ can be seen as two deep minima which lie on the $u = 0$ meridian slightly above the equatorial line and below the latitude of the A_3 points shown by a dashed line. As θ increases $\xi_\pm$ move up (north). In the region IIa, they become unstable, see case $\theta = 11\pi/36$. In the region IIb $\xi_\pm$ do not exist, they reappear for $\theta > \frac{1}{2}\pi$ as minima. Using the same local analysis as for other stationary points, i.e., finding a local symplectic chart and linearizing $\widehat{H}$ in this chart, we find that $\xi_\pm$ are always of type EH (elliptic–hyperbolic). In the regions I, IIIa, and IIIb the elliptic plane is tangent to the sphere and the hyperbolic plane is orthogonal to the sphere; in the IIa region the situation is reversed. $\square$

Remark 17. *Evolution of the additional $C_s \wedge \mathcal{T}_2$ relative equilibria can be best seen on the interval $\theta \in [-\frac{1}{2}\pi, \rho_1]$. (The part $[-\frac{1}{2}\pi, 0]$ is equivalent to region III $[\frac{1}{2}\pi, \pi]$ in fig. 2 and table 2 up to the sign of $\widehat{H}$, see remark 11.) These RE branch off the A_4 RE at $\theta = -\frac{1}{2}\pi$ and then exist continuously till their merger with the A_2 RE at $\theta = \rho_1$.*

5.2. Configuration space image of the $C_s \wedge \mathcal{T}_2$ relative equilibria

The principles of the RE representation in the configuration space $\mathbf{R}^3_{x,y,z}$ are discussed in Appendix A.3. The $C_s \wedge \mathcal{T}_2$ RE are not $\mathcal{T}$ invariant and therefore they appear in $\mathbf{R}^3$ as loops. The two $C_s \wedge \mathcal{T}_2$ stationary points on the same C_s sphere, such as, for example, in fig. 3 with $\theta = 5\pi/36$, are mapped into each other by the $\mathcal{T}$ operation. These two points correspond to two loops running along the same closed curve in $\mathbf{R}^3$ in different directions. According to remark 17, these loops branch off one of the three A_4 orbits and merge with an A_2 orbit. Take for example the three RE $A_4^{\tilde{z}}$, $A_2^{\tilde{z}}$, and $A_2^{\bar{\tilde{z}}}$. The $A_4^{\tilde{z}}$ RE is represented by a segment on axis z, while images of $A_2^{\tilde{z}}$ and $A_2^{\bar{\tilde{z}}}$ lie in the planes aOb and cOd (see fig. A1 and fig. 4, left). Note that axis z is the intersection $aOb \cap cOd$, and that the aOb and cOd planes are the configuration spaces of the restricted systems whose reduced phase spaces are the C_s^{ab} and C_s^{cd} spheres.

Without any loss to our present qualitative description, we can consider RE of the normalized system instead of those of the original system shown in fig. 1. In the transformed space $\tilde{\mathbf{R}}^3$ the $A_4^{\tilde{z}}$ RE remains a segment of axis z, while $A_2^{\tilde{z}}$ and $A_2^{\bar{\tilde{z}}}$ become segments of straight lines $x = y$ and $x = -y$ in the horizontal plane $z = 0$ as shown in fig. 4. At $\theta = \frac{1}{2}\pi$, four $C_s \wedge \mathcal{T}_2$ orbits bifurcate from $A_4^{\tilde{z}}$. These four RE project into two distinct closed curves in $\mathbf{R}^3$. When the value of θ is only slightly above $\frac{1}{2}\pi$, the loops have a highly eccentric elliptical shape stretched along $A_4^{\tilde{z}}$, see fig. 4, left. The major axis of the ellipsis is on the z axis and the ellipsis lie in the aOb and cOd planes respectively. As θ increases, the eccentricity is reduced until the ellipses become circles at $\theta = \frac{5}{4}\pi$. At this point the elliptic and hyperbolic directions

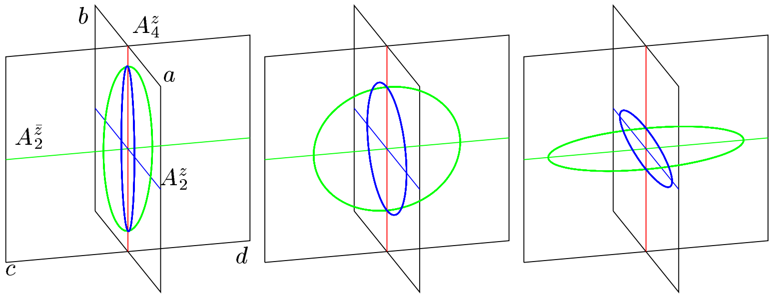

Figure 4. Schematic representation of the A_4^z, A_2^z, $A_2^{\bar{z}}$, and $C_s \wedge \mathcal{T}$ relative equilibria in the configuration space $\mathbf{R}^3_{x,y,z}$ of the normalized Hamiltonian (3).

for the $C_s \wedge \mathcal{T}_2$ points $\xi_\pm \in CP^2(n)$ are interchanged. For $\theta > \frac{5}{4}\pi$ the eccentricity increases again but now the major axes lie near the intersections the $aOb \cap xOy$ and $cOd \cap xOy$ respectively. As θ approaches $\pi + \rho_1$, the two ellipses come closer to the orbits A_2^z and $A_2^{\bar{z}}$ and vanish exactly at $\theta = \pi + \rho_1$.

6. Bifurcations

Definition 6. *The 3D Hénon-Heiles system with Hamiltonian (3) is called generic if it is ϵ^2–generic and the corresponding principal order of the reduced Hamiltonian $\widehat{H}$ in (12) is a $T_d \times \mathcal{T}$-invariant Morse function on $\mathbf{CP}^2(n)$.*

Generic systems belong to one of the subfamilies in table 2. We have charcterized the RE of these systems. In this section we comment on some of the changes of linear stability of the RE in table 2 and the possible bifurcations that may be happening when the parameter θ is varied. A full study of non-Morse members of the family (12) and bifurcations of RE requires going to higher orders of the normal form and is beyond the scope of our present work. Some of these bifurcations cannot be fully understood using the single–parameter classification scheme of sec. 2.

As we saw in sec. 5.1, several bifurcations are related to the evolution of the $C_s \wedge \mathcal{T}_2$ RE. At $\theta = \frac{1}{2}\pi$ (or $-\frac{1}{2}\pi$) four $C_s \wedge \mathcal{T}_2$ RE are created in the bifurcation of each of the three A_4 RE. In this bifurcation, as we go from the region IIb to IIIa in table 2, stability and Morse index of the A_4 RE change from 2E and 0 to 2H and 2 respectively. Since A_4 and B_4 share the same C_2–invariant subspace $\mathbf{S}^2$ (fig. A3, left), the Morse index change of A_4 *forces* the change of the Morse index of B_4 in order to preserve the right Euler characteristics of $\mathbf{S}^2$. When $\theta = \pi$ and we enter region I from IIIb, the Morse index of A_4 changes from 2 to 0 (or 4, see remark 11). When $\theta = \rho_1$ the $C_s \wedge \mathcal{T}_2$ RE collide pairwise at the A_2 RE and vanish. At this moment the A_2 RE change linear stability type from EE to EH and Morse index from 2 to 1. This bifurcation can be considered as a collision of two stationary points on the C_s sphere; in systems with one degree of freedom it is often called a “pitchfork” bifurcation.

The stability type of the B_3 points changes between elliptic-elliptic (EE) and complex hyperbolic (CH) at $\theta = \pi/4$ and $\theta = \rho_2$. At these values of θ the four eigenvalues of the respective Hamiltonian matrices move along the imaginary axis, collide pairwise, and then move off the axis into the complex plane. Such EE$\leftrightarrow$CH change of linear stability is called *linear Hamiltonian Hopf bifurcation*‡. It suggests that a *nonlinear* Hamiltonian Hopf bifurcation might also be taking place [32]. This important codimension–one bifurcation happens in Hamiltonian systems with two or more degrees of freedom. The EE↔CH change is necessary but not sufficient for the nonlinear bifurcation. The latter occurs when a family of periodic orbits either detaches from the bifurcating stationary point or shrinks to this point and disappears. We should take the nonlinearity of the system into account in order to find out if this takes place.

Unfortunately, standard theorems on the Hamiltonian Hopf bifurcation do not apply directly in our case. Thus when $\theta = \rho_2$ we can prove that the B_3 points lie on an invariant two-dimensional manifold and as a consequence, the bifurcation remains degenerate in all orders. A preliminary study using normal form techniques shows that a bifurcation of short periodic orbits which differs slightly from the Hamiltonian Hopf bifurcation takes place af $\theta = \rho_2$. At $\theta = \frac{1}{4}\pi$ the eigenvalues of the Hamiltonian matrix of the linearized equations near the B_3 equilibrium meet at 0 and then leave off to the complex plane. This means that at the moment of bifurcation the quadratic part of the local Hamiltonian is nilpotent and cannot be normalized in the standard way. We believe that both the degeneracy of the $\theta = \rho_2$ case and the nilpotency of the $\theta = \frac{1}{4}\pi$ case can be removed in the sixth (or higher) order normal form of (3).

A generalization of the linear Hamiltonian Hopf bifurcation is proposed in [33]. This paper describes a bifurcation of short periodic orbits that happens when a stationary point with isotropy group $\mathrm{SO}(2) \times \mathcal{T}$ changes linear stability type from degenerate elliptic (2E) to degenerate hyperbolic (2H). In our system, the A_3 and A_4 RE change stability type from 2E to 2H at $\theta = \frac{1}{4}\pi$ and $\theta = \frac{1}{2}\pi$ respectively. Since by theorem 3 the A_4 and A_3 RE can only be of type 2E or 2H, the eigenvalues must become simultaneously zero when their stability changes from 2E to 2H. This degeneracy is robust under $T_d \times \mathcal{T}$ symmetric perturbations.

7. Conclusion: Classification of 3D Hénon-Heiles systems

If we now consider all stationary points of $\widehat{H}$ in (12) that we have found, we can verify that Morse inequalities (B.2) are now satisfied on $\mathbf{CP}^2(n)$ and on all invariant subspaces of the $T_d \times \mathcal{T}$ action on $\mathbf{CP}^2(n)$ for all values of θ. Studying the restictions of $\widehat{H}$, we can also verify that there are no other stationary points on the subspaces. Nevertheless, this does not exclude existence of other stationary points of $\widehat{H}$.

Proposition 2. *Table 2 gives a complete set of stationary points of the reduced Hamiltonian $\widehat{H}$ in (12) in all regions of the values of the parameter θ where $\widehat{H}$ is a Morse function on $\mathbf{CP}^2$.*

Corollary 3. *The family of systems with Hamiltonian (3) and its further generalizations mentioned in remark 1 has five qualitatively different subfamilies. Each subfamily is characterized by the number and stability of nonlinear normal modes which correspond to the stationary points described in table 2. There is only one subfamily with minimal possible number (27) of nonlinear normal modes described in theorem 1. Other subfamilies have a set of* 12 *additional equivalent modes of the type $C_s \wedge \mathcal{T}_2$.*

This concludes the classification of all systems with Hamiltonian (3) generic in the sense of definition 6.

‡ The name is due to the reminiscence to the Hopf bifurcation in dissipative systems.

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Appendix A. Action of the group $T_d \times \mathcal{T}$ on the spaces $\mathbf{R}^3$, $T\mathbf{R}^3$, and $\mathbf{CP}^2$

The symmetry group $T_d \subset \mathrm{O}(3)$ of the tetrahedron is a group of point transformations of the physical 3-space, see fig. A1. As an abstract group it is isomorphic to the permutation group of 4 elements. We assume that the coordinate functions (x, y, z) in the configuration space $\mathbf{R}^3$ of (3) span a three dimensional *vector* representation of T_d, that is, T_d acts on (x, y, z) as on the coordinates in the physical 3-space. Let O

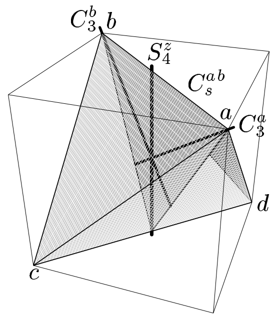


Figure A1. Symmetry axes and planes of a tetrahedron

be the origin $(0,0,0) \in \mathbf{R}^3$, and let Ox, Oy and Oz be the directed semi-axes of the coordinate system in $\mathbf{R}^3$. Consider also the 4 directed semi-axes Oa, Ob, Oc and Od in fig. A1, where $a = (1,1,1)$, $b = (-1,-1,1)$, $c = (1,-1,-1)$ and $d = (-1,1,-1)$. Any pair of semi-axes $(O\alpha, O\beta)$ defines a 2-plane $\alpha O \beta$ passing through O. Table A1 gives explicit definition of some basic operations in T_d which we further explain below.

- S_4 Operation S_4^x combines the counterclockwise rotation by $2\pi/4 = \frac{1}{2}\pi$ about Ox and the reflection in the plane $yOz \perp Ox$. Similar operations S_4^y and S_4^z involve axes Oy and Oz respectively. The conjugacy class S_4 in T_d also contains the elements $(S_4^\alpha)^{-1}$, $\alpha = x, y, z$; elements $C_2^\alpha = (S_4^\alpha)^2$ form the conjugacy class C_2.
- C_3 Operations C_3^a, C_3^b, C_3^c and C_3^d are counterclockwise rotations by $2\pi/3$ about axes Oa, Ob, Oc and Od respectively. The conjugacy class C_3 also includes $(C_3^a)^2$, $(C_3^b)^2$, $(C_3^c)^2$ and $(C_3^d)^2$.
- C_s Reflection in each of the six planes $\{aOb, cOd, aOd, bOc, aOc, bOd\}$, which we denote as C_s^{ab}, C_s^{cd}, C_s^{ad}, C_s^{bc}, C_s^{ac} and C_s^{bd}, leaves the tetrahedron invariant. These operations form the conjugacy class C_s.

Extension of the described T_d action to the phase space $T\mathbf{R}^3$ of (3) uses the lemma.

R	S_4^x	S_4^y	S_4^z	C_3^a	C_3^b	C_3^c	C_3^d	C_s^{ab}	C_s^{cd}	C_s^{ad}	C_s^{bc}	C_s^{ac}	C_s^{bd}
Rx	$-x$	z	$-y$	z	$-z$	$-z$	z	y	$-y$	z	$-z$	x	x
Ry	$-z$	$-y$	x	x	x	$-x$	$-x$	x	$-x$	y	y	z	$-z$
Rz	y	$-x$	$-z$	y	$-y$	y	$-y$	z	z	x	$-x$	y	$-y$

Table A1. Action of some elements of T_d on the vector representation spanned by the coordinate functions x, y, z in $\mathbf{R}^3$.

Lemma A.1. *If matrices M_q and M_p in $GL(\mathbf{R}, 3)$ acting on the coordinates $q = (x, y, z)$ and the conjugated momenta $p = (p_x, p_y, p_z)$ define a linear symplectic transformation in $T\mathbf{R}^3$, then $M_p = \left(M_q^{-1}\right)^T$.*

It follows that $M_q = M_p$ for $M_q \in T_d \subset O(3)$, that is, (x, y, z) and (p_x, p_y, p_z) transform according to the *same* representation of T_d. Action of the full symmetry group $T_d \times \mathcal{T}$ on $T\mathbf{R}^3$ is obtained by combining actions of T_d and the momentum reversal $\mathcal{T} : (q, p) \to (q, -p)$.

Appendix A.1. Fixed points of the action of $T_d \times \mathcal{T}$ on $\mathbf{CP}^2$

Projection of the $T_d \times \mathcal{T}$ action on $\mathbf{CP}^2$ has been discussed in detail in [19, 18, 14, 13]. We give only the information that is useful for our study. Action of $T_d \times \mathcal{T}$ on the invariants (6) can be found straightforwardly using the action of $T_d \times \mathcal{T}$ on $T\mathbf{R}^3$, table A2 illustrates the results. Zhilinskií described the critical orbits of the T_d action on $\mathbf{CP}^2$ in [19]. The action of the full group $T_d \times \mathcal{T}$ has the same 5 critical orbits [14, 13] which we characterize in table 1 and A3. Remark that points of types A and B transform differently with respect to $\mathcal{T}$: A_4, A_3, and A_2 are $\mathcal{T}$-invariant, while B_4 and B_3 are not because $\mathcal{T}$ maps each B-type point to another, for example $B_4^z \to B_4^{\bar{z}}$.

R	$R\nu_1$	$R\nu_2$	$R\nu_3$	$R\sigma_1$	$R\sigma_2$	$R\sigma_3$	$R\tau_1$	$R\tau_2$	$R\tau_3$
C_2^x	ν_1	ν_2	ν_3	σ_1	$-\sigma_2$	$-\sigma_3$	τ_1	$-\tau_2$	$-\tau_3$
S_4^x	ν_1	ν_3	ν_2	$-\sigma_1$	$-\sigma_3$	σ_2	τ_1	τ_3	$-\tau_2$
C_3^a	ν_3	ν_1	ν_2	σ_3	σ_1	σ_2	τ_3	τ_1	τ_2
C_s^{ab}	ν_2	ν_1	ν_3	σ_2	σ_1	σ_3	$-\tau_2$	$-\tau_1$	$-\tau_3$
$\mathcal{T}$	ν_1	ν_2	ν_3	σ_1	σ_2	σ_3	$-\tau_1$	$-\tau_2$	$-\tau_3$

Table A2. Action of some elements of $T_d \times \mathcal{T}$ on $\mathbf{CP}^2$

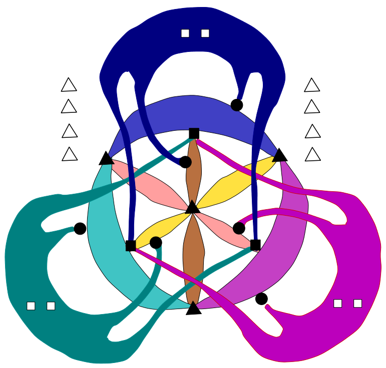
Appendix A.2. Subspaces of $\mathbf{CP}^2$ invariant under the action of $T_d \times \mathcal{T}$

The action of $T_d \times \mathcal{T}$ on $\mathbf{CP}^2$ has a number of invariant subspaces M of topology $\mathbf{RP}^2$, $\mathbf{CP}^1 \sim \mathbf{S}^2$, and $\mathbf{S}^1$ [14]. Points of M are non-isolated fixed points of the action of the stabilizer $G_M \subset T_d \times \mathcal{T}$ of M. Stabilizers C_2 and C_s of the $\mathbf{S}^2$ spheres are symplectic and the spheres remain invariant under the flow of any $T_d \times \mathcal{T}$–invariant Hamiltonian $\widehat{H}$. According to [19], the 27 critical points and the spheres intersect in $\mathbf{CP}^2$ as shown in fig. A2. We discuss these spheres in more detail.

Consider specifically the action of $C_2^x \subset T_d$ on $\mathbf{CP}^2(n)$. Using table A2 we find that fixed points of this action are of the form $(\nu_1, \nu_2, \nu_3; \sigma_1, 0, 0; \tau_1, 0, 0)$. Taking

Orbit	Point	Isotropy	
A_4	A_4^x	$\mathcal{D}_{2d}^x \times \mathcal{T}$	$\{1, S_4^x, C_2^x, (S_4^x)^{-1}, C_2^y, C_2^z, C_s^{ac}, C_s^{bd}\} \times \mathcal{T}$
	A_4^y	$\mathcal{D}_{2d}^y \times \mathcal{T}$	$\{1, S_4^y, C_2^y, (S_4^y)^{-1}, C_2^x, C_2^z, C_s^{ad}, C_s^{bc}\} \times \mathcal{T}$
	A_4^z	$\mathcal{D}_{2d}^z \times \mathcal{T}$	$\{1, S_4^z, C_2^z, (S_4^z)^{-1}, C_2^x, C_2^y, C_s^{ab}, C_s^{cd}\} \times \mathcal{T}$
A_3	A_3^a	$\mathcal{C}_{3v}^a \times \mathcal{T}$	$\{1, C_3^a, (C_3^a)^2, C_s^{ab}, C_s^{ac}, C_s^{ad}\} \times \mathcal{T}$
	A_3^b	$\mathcal{C}_{3v}^b \times \mathcal{T}$	$\{1, C_3^b, (C_3^b)^2, C_s^{ab}, C_s^{bc}, C_s^{bd}\} \times \mathcal{T}$
	A_3^c	$\mathcal{C}_{3v}^c \times \mathcal{T}$	$\{1, C_3^c, (C_3^c)^2, C_s^{ac}, C_s^{bc}, C_s^{cd}\} \times \mathcal{T}$
	A_3^d	$\mathcal{C}_{3v}^d \times \mathcal{T}$	$\{1, C_3^d, (C_3^d)^2, C_s^{ad}, C_s^{bd}, C_s^{cd}\} \times \mathcal{T}$
A_2	A_2^x	$\mathcal{C}_{2v}^x \times \mathcal{T}$	$\{1, C_2^x, C_s^{ac}, C_s^{bd}\} \times \mathcal{T}$
	$A_2^{\bar{x}}$	$\mathcal{C}_{2v}^x \times \mathcal{T}$	$\{1, C_2^x, C_s^{ac}, C_s^{bd}\} \times \mathcal{T}$
	A_2^y	$\mathcal{C}_{2v}^y \times \mathcal{T}$	$\{1, C_2^y, C_s^{ad}, C_s^{bc}\} \times \mathcal{T}$
	$A_2^{\bar{y}}$	$\mathcal{C}_{2v}^y \times \mathcal{T}$	$\{1, C_2^y, C_s^{ad}, C_s^{bc}\} \times \mathcal{T}$
	A_2^z	$\mathcal{C}_{2v}^z \times \mathcal{T}$	$\{1, C_2^z, C_s^{ab}, C_s^{cd}\} \times \mathcal{T}$
	$A_2^{\bar{z}}$	$\mathcal{C}_{2v}^z \times \mathcal{T}$	$\{1, C_2^z, C_s^{ab}, C_s^{cd}\} \times \mathcal{T}$
B_4	B_4^x	$\mathcal{S}_4^x \times \mathcal{T}_2^y$	$\{1, S_4^x, C_2^x, (S_4^x)^{-1}, C_2^y T, C_2^z T, C_s^{ac} T, C_s^{bd} T\}$
	$B_4^{\bar{x}}$	$\mathcal{S}_4^x \times \mathcal{T}_2^y$	$\{1, S_4^x, C_2^x, (S_4^x)^{-1}, C_2^y T, C_2^z T, C_s^{ac} T, C_s^{bd} T\}$
	B_4^y	$\mathcal{S}_4^y \times \mathcal{T}_2^z$	$\{1, S_4^y, C_2^y, (S_4^y)^{-1}, C_2^x T, C_2^z T, C_s^{ad} T, C_s^{bc} T\}$
	$B_4^{\bar{y}}$	$\mathcal{S}_4^y \times \mathcal{T}_2^z$	$\{1, S_4^y, C_2^y, (S_4^y)^{-1}, C_2^x T, C_2^z T, C_s^{ad} T, C_s^{bc} T\}$
	B_4^z	$\mathcal{S}_4^z \times \mathcal{T}_2^x$	$\{1, S_4^z, C_2^z, (S_4^z)^{-1}, C_2^x T, C_2^y T, C_s^{ab} T, C_s^{cd} T\}$
	$B_4^{\bar{z}}$	$\mathcal{S}_4^z \times \mathcal{T}_2^x$	$\{1, S_4^z, C_2^z, (S_4^z)^{-1}, C_2^x T, C_2^y T, C_s^{ab} T, C_s^{cd} T\}$
B_3	B_3^a	$\mathcal{C}_3^a \times \mathcal{T}_s^{ab}$	$\{1, C_3^a, (C_3^a)^2, C_s^{ab} T, C_s^{ac} T, C_s^{ad} T\}$
	$B_3^{\bar{a}}$	$\mathcal{C}_3^a \times \mathcal{T}_s^{ab}$	$\{1, C_3^a, (C_3^a)^2, C_s^{ab} T, C_s^{ac} T, C_s^{ad} T\}$
	B_3^b	$\mathcal{C}_3^b \times \mathcal{T}_s^{ab}$	$\{1, C_3^b, (C_3^b)^2, C_s^{ab} T, C_s^{bc} T, C_s^{bd} T\}$
	$B_3^{\bar{b}}$	$\mathcal{C}_3^b \times \mathcal{T}_s^{ab}$	$\{1, C_3^b, (C_3^b)^2, C_s^{ab} T, C_s^{bc} T, C_s^{bd} T\}$
	B_3^c	$\mathcal{C}_3^c \times \mathcal{T}_s^{cd}$	$\{1, C_3^c, (C_3^c)^2, C_s^{ac} T, C_s^{bc} T, C_s^{cd} T\}$
	$B_3^{\bar{c}}$	$\mathcal{C}_3^c \times \mathcal{T}_s^{cd}$	$\{1, C_3^c, (C_3^c)^2, C_s^{ac} T, C_s^{bc} T, C_s^{cd} T\}$
	B_3^d	$\mathcal{C}_3^d \times \mathcal{T}_s^{cd}$	$\{1, C_3^d, (C_3^d)^2, C_s^{ad} T, C_s^{bd} T, C_s^{cd} T\}$
	$B_3^{\bar{d}}$	$\mathcal{C}_3^d \times \mathcal{T}_s^{cd}$	$\{1, C_3^d, (C_3^d)^2, C_s^{ad} T, C_s^{bd} T, C_s^{cd} T\}$

Table A3. Isotropy groups (stabilizers) of the critical points of the $T_d \times \mathcal{T}$ action on $\mathbf{CP}^2_n$. In the second column $\mathcal{T}_2 = \{1, C_2 T\}$ and $\mathcal{T}_s = \{1, C_s T\}$.



fixed point	stabilizer in $T_d \times \mathcal{T}$
■	$\mathcal{D}_{2d} \times \mathcal{T}$
▲	$\mathcal{C}_{3v} \times \mathcal{T}$
□	$\mathcal{S}_4 \times \mathcal{T}_2$
△	$\mathcal{C}_3 \times \mathcal{T}_s$
•	$\mathcal{C}_{2v} \times \mathcal{T}$

Figure A2. Orbits of the T_d and $T_d \times \mathcal{T}$ group action on $\mathbf{CP}^2$ according to [19]. Colored areas represent three C_2-invariant and six C_s-invariant spheres.

relations (7) into account we find that the subset of $\mathbf{CP}^2(n)$ with stabilizer C_2^x is the disjoint union of A_2^x and the $\mathbf{S}^2$ sphere $(0,\nu_2,n-\nu_2;\sigma_1,0,0;\tau_1,0,0)$ with $\sigma_1^2+\tau_1^2+(2\nu_2-n)^2=n^2$. In the coordinates $u=\sigma_1 n^{-1}$, $v=\tau_1 n^{-1}$ and $w=2\nu_2 n^{-1}-1$, its equation is $u^2+v^2+w^2=1$. There are three C_2 spheres corresponding to the three axes C_2. On each sphere we find six critical points, two of type A_4, two of type A_2 and two of type B_4. Specifically, on the C_2^x sphere we find the points A_4^y, A_4^z, A_2^x, $A_2^{\bar{x}}$, B_4^x, and $B_4^{\bar{x}}$ (figure A3a).

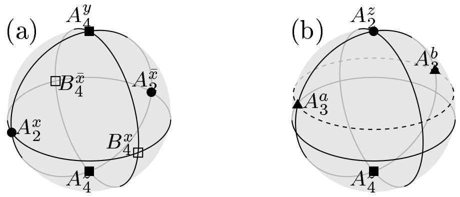


Figure A3. The C_2^x invariant sphere (a) and the C_s^{ab} invariant sphere (b) in the ambient space $\mathbf{R}^3$ with coordinates (u,v,w) adapted for each case (see text). Solid lines represent the intersections of the spheres with the planes $\{u=0\}$, $\{v=0\}$ and $\{w=0\}$. In (b) the dashed line represents the intersection of the sphere with the plane $\{w=1/3\}$.

The same analysis for the action of $C_s^{ab}\subset T_d$ shows that the set of $\mathbf{CP}^2(n)$ points fixed under this action is the disjoint union of $A_2^{\bar{z}}$ and the $\mathbf{S}^2$ sphere $(\nu_1,\nu_1,n-2\nu_1;\sigma_1,\sigma_1,2\nu_1;\tau_1,-\tau_1,0)$ with $2\sigma_1^2+2\tau_1^2+(4\nu_1-n)^2=n^2$ and coordinates $u=\sqrt{2}\sigma_1 n^{-1}$, $v=\sqrt{2}\tau_1 n^{-1}$ and $w=4\nu_1 n^{-1}-1$. There are six such spheres, one for each C_s plane. On each C_s sphere we find four critical points, one of type A_2, one of type A_4, and two of type A_3. Specifically, on the C_s^{ab} sphere we find the points A_2^z, A_4^z, A_3^a, and A_3^b (figure A3b).

Appendix A.3. Action of $T_d\times\mathcal{T}$ on the projections of nonlinear normal modes in the configuration space $\mathbf{R}^3$

Like in the 2D Hénon-Heiles system, it is quite convenient to represent the nonlinear normal modes of the 3D system with Hamiltonian (3) by their projections in the configuration space $\mathbf{R}^3_{x,y,z}$ shown in fig. 1. The qualitative "shape" of each projection can be derived from the symmetry properties (isotropy group) of the mode using a set of simple principles which we formulate below as lemmas.

Lemma A.2. *Projections $\Gamma\subset\mathbf{R}^3_{x,y,z}$ of periodic orbits of the system with Hamiltonian H in (3) can be of two types: (i) closed curves; (ii) linear (generally curved) segments or degenerate closed curves, which begin and end orthogonally at the equipotential surface, the closure of the projection of the constant level set of H in $\mathbf{R}^3$.*

Lemma A.3. *Action of the elements $g\in T_d$ on projections Γ is found straightforwardly from the action of g on each point $m\in\Gamma\subset\mathbf{R}^3$.*

Lemma A.4. *In order to study the action of time reversal operation $\mathcal{T}$ on closed curve projections Γ, we should consider the latter as directed closed curves, or loops. The two periodic orbits which project into the same closed curve Γ correspond to two loops Γ_+ and Γ_- with different directions. The $\mathcal{T}$ operation changes direction, i.e., $\mathcal{T}:\Gamma_+\leftrightarrow\Gamma_-$. A segment projection represents one $\mathcal{T}$-invariant periodic orbit.*

Lemma A.5. *Let Γ be an image of a periodic orbit defined according to lemma A.4, and $G \subset T_d \times \mathcal{T}$ be the isotropy group. Then each operation $g \in G$ maps Γ into itself as a whole, but points $m \in \Gamma$ are not necessarily fixed points of g. On the other hand, if $g \notin G$ then g defines a 1:1 map $\Gamma \to \Gamma'$ where Γ' is an image of another periodic orbit in the same group orbit.*

It follows from lemma A.4 that the $\mathcal{T}$-invariant modes A_4, A_3, and A_2 project into segments (degenerate curves). Furthermore, action of the D_{2d} stabilizers on $\mathbf{R}^3$ is such that the A_4 modes must project onto the symmetry axes (Ox, Oy, Oz), e.g., A_4^z is represented by a segment of axis Oz (see figs. 1 and A1). Similarly, the A_3 modes project onto the C_3 axes (Oa, Ob, Oc, Od). The spatial isotropy group of the A_2 modes is the group C_{2v}, whose C_2 axis is one of the (Ox, Oy, Oz). These modes project into curved line segments lying in the symmetry planes of the C_{2v} group. For example, the images of the periodic orbits A_2^z and $A_2^{\bar{z}}$ lie in the planes aOb (the plane $x = y$) and cOd (the plane $x = -y$) respectively near the intersections of these planes with the horizontal plane xOy. The images do not intersect: A_2^z passes above the xOy plane while $A_2^{\bar{z}}$ lies below it, see figure 1.

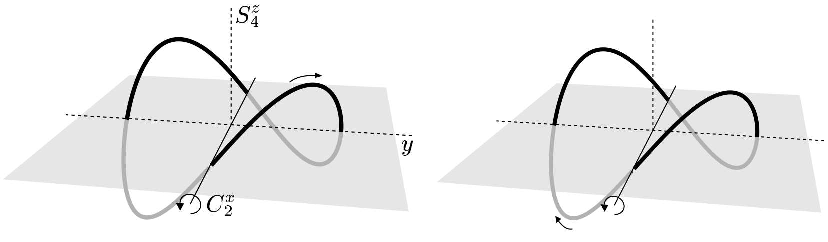


Figure A4. Two B_4^z nonlinear normal modes related by the $\mathcal{T}$ and C_2 operations. Compared to fig. 1 the z axis (vertical) scale is zoomed.

On the other hand, the modes B_3 and B_4 are not $\mathcal{T}$-invariant. They project into closed curves in fig. 1. According to lemma A.4, each such curve accomodates two loops. As an instructive example, consider the two B_4^z modes in fig. A4. In accordance with the spatial symmetry of these orbits S_4^z, their projection resembles a wobbled square whose two pairs of opposing smoothed vertexes are lifted and lowered out of the plane xOy. It is easy to see from fig. A4 that both operations C_2^x and $\mathcal{T}$ preserve this projection geometrically but change the direction of the mode, so that $B_{4+}^z \leftrightarrow B_{4-}^z$. At the same time, the modes are invariant with regard to the combination $\mathcal{T}_2 = C_2 \circ \mathcal{T}$ where $C_2 = C_2^x$ or C_2^y.

Appendix B. Local properties of relative equilibria

Appendix B.1. Morse inequalities and Euler characteristics

A function f defined on a manifold M is called a Morse function if all its stationary points $m \in M$ are nondegenerate, i.e., the determinant of the Hessian at m is not zero, $\det \partial^2 f(m) \neq 0$. The *Morse index* j of a nondegenerate stationary point m of f is defined as the number of negative eigenvalues of $\partial^2 f(m)$. Stationary points of a Morse function f must obey certain relations, called *Morse inequalities*, that are expressed in terms of the *Betti numbers*. The $\dim M + 1$ Betti numbers b_j, $j = 0, \ldots, \dim M$, are nonnegative integers that depend only on topological properties of M. These numbers and the *Euler characteristics* $B_{\dim M} = \sum_{j=0}^{\dim M} b_j$ for the spaces encountered in our work are given below.

manifold M	$\dim M$	Betti numbers	$B_{\dim M}$
$\mathbf{CP}^2$	4	$b_0 = 1,\, b_1 = 0,\, b_2 = 1,\, b_3 = 0,\, b_4 = 1$	3
$\mathbf{CP}^1 \sim \mathbf{S}^2$	2	$b_0 = 1,\, b_1 = 0,\, b_2 = 1$	2
$\mathbf{S}^1$	1	$b_0 = 1,\, b_1 = 1$	0

If c_j is the number of stationary points of f with Morse index j, and

$$\begin{aligned} C_j &= c_j - C_{j-1} \quad \text{for} \quad j = 1, \ldots, \dim M, \quad \text{and} \quad C_0 = c_0, \\ B_j &= b_j - B_{j-1} \quad \text{for} \quad j = 1, \ldots, \dim M, \quad \text{and} \quad B_0 = b_0, \end{aligned}$$

then the Morse inequalities are

$$C_j \geq B_j \quad \text{for} \quad j = 0, \ldots, n-1, \quad \text{and} \quad C_{\dim M} = B_{\dim M}. \tag{B.1}$$

Specifically, in the case of $\mathbf{CP}^2$ the inequalities (B.1) become

$$\begin{aligned} &c_0 \geq 1, \qquad c_1 - c_0 \geq -1, \qquad c_2 - c_1 + c_0 \geq 2, \\ &c_3 - c_2 + c_1 - c_0 \geq -2, \qquad c_4 - c_3 + c_2 - c_1 + c_0 = 3. \end{aligned} \tag{B.2}$$

Remark B.1. *The minimal number of stationary points of a Morse function h on $\mathbf{CP}^2$ in the absence of symmetries is three. When h has just three stationary points, Morse inequalities (B.2) become equalities and h is called* perfect *Morse function.*

Appendix B.2. Linear stability types

Consider a Hamiltonian $H : T\mathbf{R}^2 \to \mathbf{R}$ of a k degrees of freedom (K-DOF) system which has a stationary point (equilibrium) m. The spectral stability of point m is determined by the eigenvalues of the Hamiltonian matrix $\mathcal{H} \in \mathrm{sp}(k, \mathbf{R})$ of the linearized equations of motion at m. Recall that if $\lambda \in \mathbf{C}$ is one of the eigenvalues of the Hamiltonian matrix $\mathcal{H}$, then $-\lambda$ and $\bar{\lambda}$ are also eigenvalues of $\mathcal{H}$. In a 1-DOF system there are two cases, *elliptic* (E) or *stable* with two imaginary eigenvalues $\pm \mathrm{i}\omega$ and *hyperbolic* (H) or *unstable* with two real eigenvalues $\pm\lambda$. A generic equilibrium of

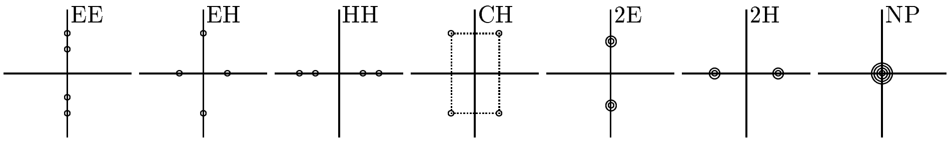


Figure B1. Types of linear stability of an equilibrium of a two degree of freedom Hamiltonian system.

a 2-DOF Hamiltonian system can be of one of the four possible linear stability types.

EE *Elliptic-Elliptic* when eigenvalues $\pm\mathrm{i}\omega_1, \pm\mathrm{i}\omega_2$, where $\omega_{1,2} \in \mathbf{R}\backslash 0$, lie on the imaginary axis. The quantities ω_1 and ω_2 are called *frequencies*.

EH *Elliptic-Hyperbolic* when two of the eigenvalues are real and two are imaginary.

HH *Hyperbolic-Hyperbolic* when all eigenvalues are real.

CH *Complex Hyperbolic* when $\lambda \neq 0$ is neither real nor imaginary and the eigenvalues form a quadruplet $\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}$.

There are also three nongeneric cases.

2E *Degenerate Elliptic* when two pairs of equal eigenvalues $(\pm \mathrm{i}\omega)$ and $(\pm \mathrm{i}\omega)$ with $\omega \in \mathbf{R} \setminus 0$ lie on the imaginary axis.

2H *Degenerate Hyperbolic* when two pairs of equal eigenvalues $(\pm\lambda)$ and $(\pm\lambda)$ with $\lambda \in \mathbf{R} \setminus 0$ lie on the real axis.

NP *Nilpotent* when all eigenvalues are zero.

All these case are illustrated in fig. B1. In a 1-DOF system the correspondence between the two stability types and the Morse index is simple: a stable point can be of index 0 or 2, while an unstable point has Morse index 1.

Lemma B.1. *In 2-DOF Hamiltonian systems we have the following relation of possible linear stability types and Morse indices.*

Morse index	0 or 4	1 or 3	2
Stability type	EE	EH	EE, HH, CH

Proof. Consider normal forms of quadratic Hamiltonians for 2-DOF systems [34] and compute possible Morse indices for each one of them. □

Appendix B.3. Linearization near stationary points on $\mathbf{CP}^2$

We explain how to compute linearized equations of motion in a local symplectic chart $T\mathbf{R}^2(\xi)$ at the stationary point $\xi \in \mathbf{CP}^2(n)$ in order to determine linear stability of ξ. Note that even though different local charts can be chosen, the linear stability type of ξ or the Morse index of ξ do not depend on the choice of coordinates.

We denote invariants in (6) as π_j, $j = 1, \ldots, 9$, and use four of these invariants as coordinates α_k, $k = 1, \ldots, 4$, in $T\mathbf{R}^2(\xi)$. If α's are chosen correctly then it should be possible to express the remaining five invariants β_l, $l = 1, \ldots, 5$ near ξ in terms of α's and n using relations Σ_i, $i = 0, \ldots, 9$ in (7). We assure this requirement by means of the implicit function theorem. We take the 9×10 Jacobian matrix $\partial\Sigma_i/\partial\pi_j$ evaluated at ξ, where $i = 0, \ldots, 9$ and $j = 1, \ldots, 9$, and select 5 rows and 5 columns of this matrix so that the determinant of the resulting 5×5 submatrix is non-zero. Invariants $\beta_1, \ldots, \beta_5$ correspond to the selected columns, and relations $\widetilde{\Sigma}_m(\beta; \alpha, n)$, $m = 1, \ldots, 5$, correspond to the selected rows; note that $\Sigma_0 \in \{\widetilde{\Sigma}\}$. We can now solve the relations $\{\widetilde{\Sigma}_m\}$ for β_l in terms of α_k and n. If the choice of $\{\beta\}$ and $\{\widetilde{\Sigma}\}$ is not unique, we aim at such choice that yields the simplest possible expressions $\beta_l(\alpha, n)$.

In order to study the system near ξ, we introduce the displacements $\delta\alpha_k$ of α_k from their values $\alpha_k(\xi)$, i.e., $\delta\alpha_k = \alpha_k - \alpha_k(\xi)$. The local coordinates $\delta\alpha = (\delta\alpha_1, \ldots, \delta\alpha_4)$ are not necessarily canonical coordinates in $T\mathbf{R}^2(\xi)$. However, it is always possible to find such linear transformation

$$(\chi, \psi) = (\chi_1, \chi_2, \psi_1, \psi_2) = B \cdot \delta\alpha$$

that the variables (χ, ψ) are canonical at $(\chi, \psi) = 0$. The Poisson brackets of these variables evaluated near $(\chi, \psi) = 0$ are $\{\chi_1, \chi_2\} = \{\psi_1, \psi_2\} = \{\chi_1, \psi_2\} = \{\chi_2, \psi_1\} = \mathcal{O}(\chi, \psi)$ and $\{\chi_1, \psi_1\} = \{\chi_2, \psi_2\} = 1 + \mathcal{O}(\chi, \psi)$.

Appendix C. Classification of the 2D Hénon-Heiles systems

Consider the 2D Hénon-Heiles system with Hamiltonian (2). The finite symmetry group of this system is $D_3 \times \mathcal{T}$, where D_3 is the dihedral group of transformations of the configuration 2-plane of (2) with coordinates (x, y). Near the linearization limit $\epsilon \to 0$ we can normalize (2) with regard to the Hamiltonian

$$H_0 = \tfrac{1}{2}(\bar{z}_x z_x + \bar{z}_y z_y)$$

of the 1:1 resonant harmonic oscillator. The reduced system has the Poisson algebra so(3) generated by the three quadratic invariants of the respective $\mathbf{S}^1$ action

$$\mathbf{j} = \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix} = \tfrac{1}{4}\begin{pmatrix} z_x\bar{z}_y + z_y\bar{z}_x \\ iz_x\bar{z}_y - iz_y\bar{z}_x \\ z_x\bar{z}_x - z_y\bar{z}_y \end{pmatrix}, \tag{C.1}$$

with Casimir $H_0 = 2j$. For all $j > 0$ the reduced phase space is the 2-sphere $\mathbf{S}^2_j \sim \mathbf{CP}^1$ of radius j defined by the equation $j_1^2 + j_2^2 + j_3^2 = j^2$ in the ambient space $\mathbf{R}^3$ with coordinates (j_1, j_2, j_3), see [35, 36].

Symmetrization of (C.1) with respect to the $D_3 \times \mathcal{T}$ action is described by the generating function

$$g(\lambda) = 1/\big[(1-\lambda^2)(1-\lambda^3)\big],$$

where λ stands for any angular momentum component in (j_1, j_2, j_3). The ring of all $\mathbf{S}^1 \times D_3 \times \mathcal{T}$ polynomials is generated *freely* by j and two other invariants which can be chosen as

$$\mu = j^2 - (j_1^2 + j_3^2) = j_2^2, \quad \xi = \tfrac{1}{2}j_3(3j_1^2 - j_3^2). \tag{C.2}$$

It follows that the normal form of (2) can be expressed as

$$\widetilde{H}_\epsilon = 2j + \epsilon^2(c_m\mu + c_0 j^2) + \epsilon^4(c_3\xi + c_{0m}\mu j + c_{00}j^3) + \mathcal{O}(\epsilon^5) \tag{C.3}$$

with constants $c_m, c_0, c_3, c_{0m}, c_{00}$. Note that μ is SO(2) symmetric and we *must* go to order ϵ^4 in order to reproduce correctly the symmetry of the 2D Hénon-Heiles system. Rescaling (C.3) by ϵ^2 and dropping constants and terms which depend only on j, we obtain the principal terms of the reduced Hamiltonian

$$\widehat{H}_\epsilon : \mathbf{S}^2_j \to \mathbf{R}, \quad \widehat{H}_\epsilon = K_m\,\mu + \epsilon^2 K_3\,\xi\ , \tag{C.4a}$$

where K_m and K_3 are constants of the same order which can always be expressed using $(c_m, c_0, c_3, c_{0m}, c_{00})$ and parameters (j, ϵ). In the simplest case of (2) with "classic" Hénon-Heiles cubic potential scaled by ϵ these constants are fixed. Since (C.4a) is not a homogeneous polynomial in $(z, \bar{z})$, the analysis for $j > 0$ is simplified if we parameterize (C.4a) after normalizing μ and ξ by j so that

$$\widehat{H} = K_m\,j^2\frac{\mu}{j^2} + \epsilon^2 K_3\,j^3\frac{\xi}{j^3} = \cos\alpha\frac{\mu}{j^2} + \sin\alpha\frac{\xi}{j^3}, \quad \tan\alpha = \epsilon^2 j\frac{K_3}{K_m}. \tag{C.4b}$$

[In other words, we project on a sphere of radius 1.] The obvious important difference from the 3D Hénon-Heiles system studied in the main body of this paper is that here α depends on the ratio of coefficients in front of the terms of *different orders*, and is therefore determined by the perturbation smallness parameter ϵ and the value of the dynamical parameter j. Thus a direct normal form computation for (2) gives

$$\alpha = \tan^{-1}\big[16\epsilon^2 j/(6 - \epsilon^2 j)\big].$$

At first sight this suggests that even though we have no external parameters in the system, we can move between different members of the family (C.4b) by increasing j. In reality, however, $\epsilon^2 j$ should be small. In fact, we can see from (C.3) that values larger than $\epsilon^2 j \approx \frac{1}{12}$ give energies above the saddle point energy $(6\epsilon^2)^{-1}$ of the potential in (2). This means that physical values of α are $0 \le \alpha \lesssim 0.07\pi$.

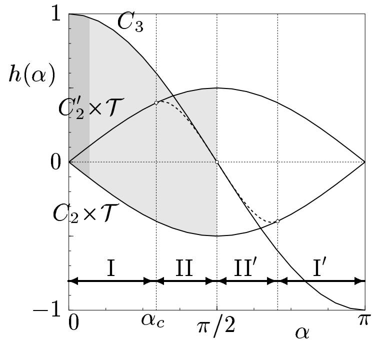


Figure C1. Energy h of the stationary points of the reduced Hamiltonian $\widehat{H}$ in (C.4b) as function of parameter α. The dashed curve marks the energy of additional stationary points with symmetry C_2; the darkest shadowed area represents reduced energies of physically possible 2D Hénon-Heiles systems with Hamiltonian (2); $\alpha_c = \tan^{-1}\frac{4}{3}$

Analysis of relative equilibria of the 2D Hénon-Heiles systems is quite similar to the one we conducted in this paper. Of course this analysis is simpler since it deals with an action of a smaller group $D_3\times\mathcal{T}$ on a smaller phase space $\mathbf{S}^2_j$. Results are well known [9, 8, 6]. At small α (C.4b) is a $D_3\times\mathcal{T}$ invariant function on $\mathbf{S}^2_j$ of the simplest kind with eight stationary points in three different critical orbits characterized below.

Traditional notation	Conjugacy class of stabilizers	values of μ	ξ	Linear stability
$\Pi_{1,2,3}$	$C_2\times\mathcal{T}$	0	$j^3/2$	H (+−)
$\Pi_{3,4,5}$	$C_2'\times\mathcal{T}$	0	$-j^3/2$	E (++)
$\Pi_{7,8}$	C_3	j^2	0	E (−−)

These points correspond to the eight nonlinear normal modes of the 2D Hénon-Heiles system with Hamiltonian (2). All such systems are of this simplest type. Extending the analysis of (C.4b) to non-physical values of α (see fig. C1) we find that on the interval $(\alpha_c,\ \pi-\alpha_c)$ this function has six extra stationary points with stabilizer C_2. As we increase α, these new points appear pairwise in the "pichfork" bifurcation with broken $\mathcal{T}$ symmetry of the three points $\Pi_{3,4,5}$ which happens when $\alpha=\alpha_c=\tan^{-1}\frac{4}{3}$. Then the new points participate in a tree-fold "touch–and–go" bifurcation of $\Pi_{7,8}$ at $\alpha=\frac{\pi}{2}$, and disappear in a pichfork bifurcation of $\Pi_{1,2,3}$ at $\alpha=\pi-\alpha_c$. There are, therefore, four qualitatively different classes of reduced systems with Hamiltonian (C.4b). These classes are shown in fig. C1; only class I can be realized in systems with Hamiltonian (2).

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